

THE EVANS-KRYLOV THEOREM FOR NONLOCAL PARABOLIC FULLY NONLINEAR EQUATIONS

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ABSTRACT. In this paper, we prove the Evans-Krylov theorem for nonlocal parabolic fully nonlinear equations.

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1. INTRODUCTION

L. Evans and N. Krylov proved independently an interior regularity for elliptic partial differential equations which states that any solution $u \in C^2(B_1)$ of a uniformly elliptic and fully nonlinear concave equation $F(D^2u) = 0$ in the unit ball $B_1 \subset \mathbb{R}^n$ satisfies an interior estimate $\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C \|u\|_{C^{1,1}(B_1)}$ with some universal constants $C > 0$ and $\alpha \in (0, 1)$, so-called the *Evans-Krylov* theorem (see [Ev], [Kr] and [CS2]). Recently, L. Caffarelli and L. Silvestre [CS1] proved a nonlocal elliptic version of the Evans-Krylov theorem which describes that any viscosity solution $u \in L^\infty(\mathbb{R}^n)$ of concave homogeneous equation on $B_1 \subset \mathbb{R}^n$ formulated by elliptic integro-differential operators of order $\sigma \in (0, 2)$ satisfies an estimate $\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}$ with universal constants $C > 0$ and $\alpha \in (0, 1)$. This nonlocal result makes it possible to recover the Evans-Krylov theorem as $\sigma \rightarrow 2^-$. In this paper, we prove a parabolic version of the nonlocal elliptic result of Caffarelli and Silvestre.

We consider the linear *parabolic integro-differential operators* given by

$$(1.1) \quad Lu(x, t) - \partial_t u(x, t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) dy - \partial_t u(x, t)$$

for $\mu_t(u, x, y) = u(x + y, t) + u(x - y, t) - 2u(x, t)$. Here we write $\mu(u, x, y) = u(x + y) + u(x - y) - 2u(x)$ if u is independent of t . We refer the detailed definitions

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of notations to [CS1, KL1, KL2, KL3]. Then we see that $Lu(x, t)$ is well-defined provided that $u \in C_x^{1,1}(x, t) \cap B(\mathbb{R}_T^n)$ where $B(\mathbb{R}_T^n)$ denotes the family of all real-valued bounded functions defined on $\mathbb{R}_T^n := \mathbb{R}^n \times (-T, 0]$ and $C_x^{1,1}(x, t)$ means $C^{1,1}$ -function in x -variable at a given point (x, t) . Moreover, $Lu(x, t)$ is well-defined even for $u \in C_x^{1,1}(x, t) \cap L_T^\infty(L_\omega^1)$ (see [KL4]).

We say that the operator L belongs to $\mathfrak{L}_0 = \mathfrak{L}_0(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_0 = \mathcal{K}_0(\sigma)$ satisfies the uniform ellipticity assumption:

$$(1.2) \quad (2 - \sigma) \frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2 - \sigma) \frac{\Lambda}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2.$$

If $K(y) = c_{n,\sigma}|y|^{-n-\sigma}$ where $c_{n,\sigma} > 0$ is the normalization constant comparable to $\sigma(2 - \sigma)$ given by

$$c_{n,\sigma} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(y_1)}{|y|^{n+\sigma}} dy \right)^{-1},$$

then the corresponding operator is $L = -(-\Delta)^{\sigma/2}$. Also we say the operator $L \in \mathfrak{L}_0$ belongs to $\mathfrak{L}_1 = \mathfrak{L}_1(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_1 = \mathcal{K}_1(\sigma)$ satisfies $K \in C^1$ away from the origin and satisfies

$$(1.3) \quad |\nabla K(y)| \leq \frac{C}{|y|^{n+1+\sigma}}.$$

Finally we say that the operator $L \in \mathfrak{L}_1$ belongs to $\mathfrak{L}_2 = \mathfrak{L}_2(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_2 = \mathcal{K}_2(\sigma)$ satisfies $K \in C^2$ away from the origin and satisfies

$$(1.4) \quad |D^2 K(y)| \leq \frac{C}{|y|^{n+2+\sigma}}.$$

The maximal operators are defined by

$$\begin{aligned} \mathbf{M}_0^+ u(x, t) &= \sup_{L \in \mathfrak{L}_0} Lu(x, t) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \mu_t^+(u, x, y) - \lambda \mu_t^-(u, x, y)}{|y|^{n+\sigma}} dy, \\ \mathbf{M}_1^+ u(x, t) &= \sup_{L \in \mathfrak{L}_1} Lu(x, t) \quad \text{and} \quad \mathbf{M}_2^+ u(x, t) = \sup_{L \in \mathfrak{L}_2} Lu(x, t). \end{aligned}$$

We shall consider nonlinear integro-differential operators, which originates from stochastic control theory with jump processes related with

$$\mathbf{I}u(x, t) = \inf_{\beta \in \mathcal{B}} L_\beta u(x, t),$$

where $L_\beta u(x, t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u, x, y) K_\beta(y) dy$ (see [AK, CS1, KL1, KL2, MP, MR] for the elliptic case and [KL3, KL4] for the parabolic case). In this paper, we are mainly interested in the nonlocal parabolic concave equations

$$(1.5) \quad \mathbf{I}u(x, t) - \partial_t u(x, t) = 0 \quad \text{in } Q_1.$$

[Notations and Definitions] Let $\sigma \in (0, 2)$ and $r > 0$.

- Denote by $Q_r = B_r \times I_r^\sigma$ and $Q_r(x, t) = Q_r + (x, t)$ for $(x, t) \in \mathbb{R}_T^n$, where $B_r(x)$ is the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, $B_r = B_r(0)$ and $I_r^\sigma = (-r^\sigma, 0]$.
- For a bounded domain $\Omega \subset \mathbb{R}^n$ and $\tau \in (0, T)$, we denote the parabolic boundary of $\Omega_\tau = \Omega \times (-\tau, 0]$ by $\partial_p \Omega_\tau := \partial_x \Omega_\tau \cup \partial_b \Omega_\tau := \partial \Omega \times (-\tau, 0] \cup \Omega \times \{-\tau\}$.

- The parabolic distance d between $X = (x, t)$ and $Y = (y, s)$ is defined by

$$(1.6) \quad d(X, Y) = \begin{cases} (|x - y|^\sigma + |t - s|)^{1/\sigma}, & t \leq s, \\ \infty, & t > s. \end{cases}$$

For $X_0 = (x_0, t_0) \in \mathbb{R}_T^n$, we set $B_r^d(x_0, t_0) = \{(x, t) \in \mathbb{R}_T^n : d(X, X_0) < r\}$.

- We denote by $\omega_\sigma(y) = 1/(1 + |y|^{n+\sigma})$ and $\omega := \omega_{\sigma_0}$ for some $\sigma_0 \in (1, 2)$ very close to 1, and also we denote by $\omega(B_{r/2}) = \int_{B_{r/2}} \omega(y) dy$. Let $\mathfrak{F}(\mathbb{R}_T^n)$ be the family of all real-valued measurable functions defined on $\mathbb{R}_T^n := \mathbb{R}^n \times (-T, 0]$. For $u \in \mathfrak{F}(\mathbb{R}_T^n)$ and $t \in (-T, 0]$, we define the weighted norm $\|u(\cdot, t)\|_{L_\omega^1}$ by

$$\|u(\cdot, t)\|_{L_\omega^1} = \int_{\mathbb{R}^n} |u(x, t)| \omega(x) dx.$$

Consider the function space $L_T^\infty(L_\omega^1)$ of all continuous L_ω^1 -valued functions $u \in \mathfrak{F}(\mathbb{R}_T^n)$ given by the family

$$\left\{ u \in \mathfrak{F}(\mathbb{R}_T^n) : \|u\|_{L_T^\infty(L_\omega^1)} < \infty, \lim_{s \rightarrow t^-} \|u(\cdot, s) - u(\cdot, t)\|_{L_\omega^1} = 0 \ \forall t \in (-T, 0] \right\}$$

with the norm $\|u\|_{L_T^\infty(L_\omega^1)} = \sup_{t \in (-T, 0]} \|u(\cdot, t)\|_{L_\omega^1}$, which is separable with

respect to the topology given by the norm.

- A mapping $\mathbb{I} : \mathfrak{F}(\mathbb{R}_T^n) \rightarrow \mathfrak{F}(\mathbb{R}_T^n)$ given by $u \mapsto \mathbb{I}u$ is called a *nonlocal parabolic operator* if (a) $\mathbb{I}u(x, t)$ is well-defined for any $u \in C_x^2(x, t) \cap L_T^\infty(L_\omega^1)$ and (b) $\mathbb{I}u$ is continuous on $\Omega_\tau \subset \mathbb{R}_T^n$, whenever $u \in C_x^2(\Omega_\tau) \cap L_T^\infty(L_\omega^1)$, where $C_x^2(x, t)$ is the class of all $u \in \mathfrak{F}$ whose second derivatives D^2u in space variables exist at (x, t) and $C_x^2(\Omega_\tau)$ denotes the class of all $u \in \mathfrak{F}$ such that $u \in C_x^2(x, t)$ for any $(x, t) \in \Omega_\tau$ and $\sup_{(x, t) \in \Omega_\tau} |D^2u(x, t)| < \infty$. Such a nonlocal operator \mathbb{I} is said to be *uniformly elliptic* with respect to a class \mathfrak{L} of linear integro-differential operators if

$$(1.7) \quad \mathbf{M}_{\mathfrak{L}}^- v(x, t) \leq \mathbb{I}(u + v)(x, t) - \mathbb{I}u(x, t) \leq \mathbf{M}_{\mathfrak{L}}^+ v(x, t)$$

where $\mathbf{M}_{\mathfrak{L}}^- v(x, t) := \inf_{L \in \mathfrak{L}} Lv(x, t)$ and $\mathbf{M}_{\mathfrak{L}}^+ v(x, t) := \sup_{L \in \mathfrak{L}} Lv(x, t)$.

- For $u \in C(Q_r)$, we define $\|u\|_{C(Q_r)} = \sup_{(x, t) \in Q_r} |u(x, t)|$. For $\alpha \in (0, 1]$ and $\sigma \in (0, 2)$, we define the *parabolic α^{th} Hölder seminorm* of u by

$$[u]_{C^\alpha(Q_r)} = \sup_{(x, t), (y, s) \in Q_r} \frac{|u(x, t) - u(y, s)|}{(|x - y|^\sigma + |t - s|)^{\alpha/\sigma}}.$$

In particular, if $0 < \alpha/\sigma < 1$, then we define the norm

$$(1.8) \quad \begin{aligned} \|u\|_{C^{\sigma+\alpha}(Q_r)} &= \|u\|_{C(Q_r)} + \|\partial_t u\|_{C(Q_r)} + \|(-\Delta)^{\sigma/2} u\|_{C(Q_r)} \\ &\quad + \|(Du) \mathbb{1}_{[1, 2)}(\sigma)\|_{C(Q_r)} + [\partial_t u]_{C^\alpha(Q_r)} + [(-\Delta)^{\sigma/2} u]_{C^\alpha(Q_r)}. \end{aligned}$$

- Let $f : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ be a continuous function and let $J := (a, b] \subset I := (-T, 0]$. Then a function $u : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ being upper (lower) semicontinuous on $\bar{\Omega} \times J$ is said to be a *viscosity subsolution* (res. *viscosity supersolution*) of an equation $\mathbb{I}u - \partial_t u = f$ on $\Omega \times J$ and we write $\mathbb{I}u - \partial_t u \geq f$ (res. $\mathbb{I}u - \partial_t u \leq f$) on $\Omega \times J$ in the viscosity sense, if for any $(x, t) \in \Omega \times J$ there is a neighborhood $Q_r(x, t) \subset \Omega \times J$ of (x, t) such that $\mathbb{I}v(x, t) - \partial_t \varphi(x, t)$ is well-defined and $\mathbb{I}v(x, t) - \partial_t \varphi(x, t) \geq f(x, t)$ (res. $\mathbb{I}v(x, t) - \partial_t \varphi(x, t) \leq f(x, t)$) for $v = \varphi \mathbb{1}_{Q_r(x, t)} + u \mathbb{1}_{Q_r^c(x, t)}$ whenever $\varphi \in C^2(Q_r(x, t))$ with $\varphi(x, t) =$

$u(x, t)$ and $\varphi > u$ ($\varphi < u$) on $Q_r(x, t) \setminus \{(x, t)\}$ exists. Here, we denote such a function φ by $\varphi \in C_{\Omega \times J}^2(u; x, t)^+$ (res. $\varphi \in C_{\Omega \times J}^2(u; x, t)^-$). Also a function u is called as a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution to $\mathbb{I}u - \partial_t u = f$ on $\Omega \times J$ (see [KL3, KL4]).

- For $a, b \in \mathbb{R}$, we denote by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.
- For a multiindex $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$, we denote by $|\beta| = \sum_{i=1}^n \beta_i$.
- Throughout this paper, let $\eta \in (0, 1)$ be a fixed sufficiently small positive number.
- For two quantities a and b , we write $a \lesssim b$ (resp. $a \gtrsim b$) if there is a universal constant $C > 0$ (depending only on $\lambda, \Lambda, n, \eta, \sigma_0$ and the constants in (1.3), (1.4) and (2.2), but not on σ) such that $a \leq Cb$ (resp. $b \leq Ca$).
- For Q_r , we denote by $C^2(Q_r) = C_x^2(Q_r) \cap C_t^1(Q_r)$ the class of functions $u \in \mathfrak{F}(\mathbb{R}^n)$ which is C^2 in space and C^1 in time on Q_r .
- For $(z, s) \in \mathbb{R}_T^n$ and $u \in \mathfrak{F}(\mathbb{R}_T^n)$, we denote the translation operators τ_z, τ^s and τ_z^s by $\tau_z u(x, t) = u(x + z, t)$, $\tau^s u(x, t) = u(x, t + s)$ and $\tau_z^s u(x, t) = u(x + z, t + s)$, respectively.

We shall now state the main theorem. The following $C^{\sigma+\alpha}$ -estimate for nonlocal parabolic concave equation for $\sigma + \alpha \geq 2$ and $\sigma \in (1, 2)$ makes it possible to recover the well-known Evans-Krylov estimate as $\sigma \rightarrow 2^-$. If $\sigma + \alpha < 2$, then $C^{\sigma+\alpha}$ -estimate is covered by $C^{1,\beta}$ -estimate in [KL3]. Our proof of the main theorem is based on the nonlocal elliptic results of Silvestre and Caffarelli [CS1] and the regularity results on nonlocal parabolic equations [KL3, KL4].

Theorem 1.1. *Let $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution of the concave equation*

$$\mathbf{I}u - \partial_t u = 0 \text{ in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (0, 2)$ as in (1.5). Then there exists a constant $\alpha \in (0, \frac{1}{4} \wedge \sigma_0 \wedge |\sigma_0 - 1|)$ such that

$$\|u\|_{C^{\sigma+\alpha}(Q_{1/2})} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Remark. (i) As mentioned above, given any $\sigma_0 \in (1, 2)$ very close to 1, it suffices to prove this theorem only for $\sigma + \alpha \geq 2$ and $\sigma \in [\sigma_0, 2)$.

(ii) In fact, from p.1569 of [KL3] and (i) we could select such $\alpha > 0$ so that $\alpha \in (0, \frac{1}{4} \wedge \sigma_0 \wedge |\sigma_0 - 1|)$ in the above. This implies that $0 < \alpha < 2 + \alpha - \sigma < 1$.

2. PARABOLIC INTERPOLATION INEQUALITIES

Let $u \in C(Q_r)$. For $0 < \alpha \leq 1$ and $\sigma \in (0, 2)$, we define the α^{th} Hölder seminorms of u in the space and time variable, respectively;

$$\begin{aligned} \text{(i)} \quad [u]_{C_x^\alpha(Q_r)} &= \sup_{t \in (-r^\sigma, 0]} \sup_{(x,t), (y,t) \in Q_r} \frac{|u(x,t) - u(y,t)|}{|x - y|^\alpha}, \\ \text{(ii)} \quad [u]_{C_t^\alpha(Q_r)} &= \sup_{x \in B_r} \sup_{(x,t), (x,s) \in Q_r} \frac{|u(x,t) - u(x,s)|}{|t - s|^\alpha}. \end{aligned}$$

If $0 < \alpha/\sigma \leq 1$, then it is easy to check that the seminorms $[\cdot]_{C_x^\alpha(Q_r)} + [\cdot]_{C_t^{\frac{\alpha}{\sigma}}(Q_r)}$ and $[\cdot]_{C^\alpha(Q_r)}$ are equivalent.

We furnish an useful parabolic interpolation inequalities which simplify the proof of our main.

Theorem 2.1. *If $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution of the concave equation*

$$\mathbf{I}u - \partial_t u = 0 \text{ in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$ very close to 1, then there exists a constant $\alpha \in (0, 1)$ with $\sigma + \alpha \geq 2$ such that

$$\|u\|_{C(Q_r)} \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \quad \text{and} \quad \|(-\Delta)^{\sigma/2} u\|_{C(Q_r)} \vee \|\partial_t u\|_{C(Q_r)} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $r \in (0, 2)$.

Proof. By rescaling, the first and second inequalities can be shown as in Theorem 5.1 and Corollary 7.4 below, respectively (also, refer to [KL4]). \square

Remark. (i) The main idea for the proof of the first inequality comes from that of parabolic Harnack inequality, and so it still holds without the concavity of the equation (see [KL3]).

(ii) Since $0 < \alpha/\sigma < 1$, this theorem and (1.8) imply that we have only to control the seminorms $[\partial_t u]_{C^\alpha(Q_r)}$ and $[(-\Delta)^{\sigma/2} u]_{C^\alpha(Q_r)}$ in order to control the norm $\|u\|_{C^{\sigma+\alpha}(Q_r)}$.

Next we give a fundamental lemma which facilitates the proof of another type of parabolic interpolation inequalities.

Lemma 2.2. *If $u \in L_T^\infty(L_\omega^1)$ is a function with $u(\cdot, t) \in C^k(B_r)$ for $t \in (-r^\sigma, 0]$ and $[D^\beta u]_{C_x^\alpha(Q_r)} < \infty$ for some $\alpha \in (0, 1)$, then for each $t \in (-r^\sigma, 0]$ and multiindex β with $|\beta| = k \in \mathbb{N}$, there exists some $z_0^t \in B_r$ (depending on t) such that*

$$|D^\beta u(z_0^t, t)| \leq \left(\frac{3r}{2}\right)^\alpha [D^\beta u]_{C_x^\alpha(Q_r)} + \frac{2(4k)^k}{\omega(B_{r/2}) r^k} \|u\|_{L_T^\infty(L_\omega^1)}.$$

Proof. Take $h = \frac{r}{2^k}$ and any multiindex β with $|\beta| = k$. For $(y, t) \in B_{r/2} \times (-T, 0]$, we consider the finite difference operator $D_h^\beta u(y, t) = D_{h,1}^{\beta_1} D_{h,2}^{\beta_2} \cdots D_{h,n}^{\beta_n} u(y, t)$ where

$$D_{h,i} u(y, t) = \frac{1}{h} [u(y + h e_i, t) - u(y, t)]$$

for a standard basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . For $i = 1, \dots, n$, we observe that

$$(2.1) \quad D_{h,i}^{\beta_i} u(y, t) = \frac{1}{h^{\beta_i}} \sum_{s=0}^{\beta_i} (-1)^s \frac{\beta_i!}{(\beta_i - s)! s!} u(y + (\beta_i - s) h e_i, t).$$

By the mean value theorem, we see that there are some $z_1^t \in B_h(y)$ and $z_2^t \in B_{2h}(y)$ such that

$$D_{h,i} D_{h,j} u(y, t) = \partial_{y_i} [D_{h,j} u](z_1^t, t) = D_{h,j} (\partial_{y_i} u)(z_1^t, t) = \partial_{y_i y_j} u(z_2^t, t).$$

This implies that $D_h^\beta u(y, t) = D^\beta u(z_y^t, t)$ for some $z_y^t \in B_{r/2}(y)$. Thus it follows from this and (2.1) that

$$\begin{aligned} \omega(B_{r/2}) |D^\beta u(z_0^t, t)| &\leq \left| \omega(B_{r/2}) D^\beta u(z_0^t, t) - \int_{\mathbb{R}^n} D_h^\beta u(y, t) \omega(y) dy \right| + \frac{2^k}{h^k} \|u\|_{L_T^\infty(L_\omega^1)} \\ &\leq \int_{B_{r/2}} |D^\beta u(z_0^t, t) - D^\beta u(z_y^t, t)| \omega(y) dy + \frac{2^{k+1}}{h^k} \|u\|_{L_T^\infty(L_\omega^1)} \\ &\leq [D^\beta u]_{C_x^\alpha(Q_r)} \left(\frac{3r}{2}\right)^\alpha \omega(B_{r/2}) + \frac{2(4k)^k}{r^k} \|u\|_{L_T^\infty(L_\omega^1)}. \end{aligned}$$

Therefore, this completes the proof. \square

Theorem 2.3. *If $u \in L_T^\infty(L_\omega^1)$ is a function such that $u(\cdot, t) \in C^k(B_r)$ for each $t \in (-r^\sigma, 0]$ and $[D^\beta u]_{C_x^\alpha(Q_r)} < \infty$ for some $\alpha \in (0, 1)$, then we have that*

$$\|D^\beta u\|_{C(Q_r)} \leq 2 \left(\frac{3r}{2}\right)^\alpha [D^\beta u]_{C_x^\alpha(Q_r)} + \frac{2(4k)^k}{\omega(B_{r/2}) r^k} \|u\|_{L_T^\infty(L_\omega^1)}$$

for any multiindex β with $|\beta| = k \in \mathbb{N}$.

Proof. From Lemma 2.2, for any $(x, t) \in Q_r$ we obtain that

$$\begin{aligned} |D^\beta u(x, t)| &\leq |D^\beta u(x, t) - D^\beta u(z_0^t, t)| + |D^\beta u(z_0^t, t)| \\ &\leq 2 [D^\beta u]_{C_x^\alpha(Q_r)} \left(\frac{3r}{2}\right)^\alpha + \frac{2(4k)^k}{\omega(B_{r/2}) r^k} \|u\|_{L_T^\infty(L_\omega^1)}. \end{aligned}$$

Hence we can have the required inequality. \square

In order to understand the **parabolic Hölder spaces** $C^{k, \gamma}(Q_r)$ with $k \in \mathbb{N}$ and $\gamma \in (0, 1)$, we define the Hölder spaces $C_x^{k, \gamma}(Q_r)$ and $C_t^{k, \gamma}(Q_r)$ in the space and time variable, respectively. For $u \in C(Q_r)$, we define the norms

$$\begin{aligned} \|u\|_{C_x^{k, \gamma}(Q_r)} &= \|u\|_{C(Q_r)} + \sum_{i=1}^k \|D^i u\|_{C(Q_r)} + [D^k u]_{C_x^\gamma(Q_r)}, \\ \|u\|_{C_t^{k, \gamma}(Q_r)} &= \|u\|_{C(Q_r)} + \sum_{i=1}^k \|\partial_t^i u\|_{C(Q_r)} + [\partial_t^k u]_{C_t^\gamma(Q_r)}, \end{aligned}$$

where $\|D^i u\|_{C(Q_r)} = \sum_{|\beta|=i} \|D^\beta u\|_{C(Q_r)}$ and $[D^k u]_{C_x^\gamma(Q_r)} = \sum_{|\beta|=k} [D^\beta u]_{C_x^\gamma(Q_r)}$ for $i, k \in \mathbb{N}$. And we denote by $C_x^{k, \gamma}(Q_r) = \{u \in \mathfrak{F}(\mathbb{R}_T^n) : \|u\|_{C_x^{k, \gamma}(Q_r)} < \infty\}$ and $C_t^{k, \gamma}(Q_r) = \{u \in \mathfrak{F}(\mathbb{R}_T^n) : \|u\|_{C_t^{k, \gamma}(Q_r)} < \infty\}$.

If $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$ and $\alpha \in (0, \sigma_0 - 1)$, then $0 < \alpha < 2 + \alpha - \sigma < 1$ and

$$\frac{2 + \alpha - \sigma}{\sigma} + 1 = \frac{2 + \alpha}{\sigma}.$$

Then we define the **parabolic Hölder space** $C^{2, \alpha}(Q_r)$ endowed with the norm

$$\begin{aligned} \|u\|_{C^{2, \alpha}(Q_r)} &= \|u\|_{C(Q_r)} + \sum_{i=1}^2 \|D^i u\|_{C(Q_r)} + \|\partial_t u\|_{C(Q_r)} \\ &\quad + [D^2 u]_{C^\alpha(Q_r)} + [\partial_t u]_{C^{2+\alpha-\sigma}(Q_r)}. \end{aligned}$$

In the same case as the above, we can learn from Theorem 2.1 and Theorem 2.3 that the estimates on the norm $\|u\|_{C^{2, \alpha}(Q_r)}$ must be controlled by those on the seminorms $[\partial_t u]_{C^{2+\alpha-\sigma}(Q_r)} \sim [\partial_t u]_{C_x^{2+\alpha-\sigma}(Q_r)} + [\partial_t u]_{C_t^{\frac{2+\alpha-\sigma}{\sigma}}(Q_r)}$ and $[D^2 u]_{C^\alpha(Q_r)} \sim [D^2 u]_{C_x^\alpha(Q_r)} + [D^2 u]_{C_t^{\frac{\alpha}{\sigma}}(Q_r)}$. Similarly, the other parabolic Hölder spaces can be defined along this line.

Lemma 2.4. *Let $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$ and $\alpha \in (0, \sigma_0 - 1)$. If $u \in L_T^\infty(L_\omega^1)$ is a function with $u(x, \cdot) \in C^1(-r^\sigma, 0]$ for $x \in B_r$ and $[\partial_t u]_{C_t^{\frac{2+\alpha-\sigma}{\sigma}}(Q_r)} < \infty$, then we have that*

$$\|\partial_t u\|_{C(Q_r)} \leq r^{2+\alpha-\sigma} [\partial_t u]_{C_t^{\frac{2+\alpha-\sigma}{\sigma}}(Q_r)} + \frac{4}{r^\sigma} \|u\|_{C(Q_r)}.$$

Proof. Take any $r \in (0, 2)$ and $(x, t) \in Q_r$. Then there is some $t_0 \in (-r^\sigma, 0]$ such that $|t - t_0| = \frac{1}{2}r^\sigma$, and by the mean value theorem, there is some t_0^x between t and t_0 such that $u(x, t_0) - u(x, t) = \frac{1}{2}r^\sigma \partial_t u(x, t_0^x)$. Thus we have the estimate

$$\begin{aligned} \frac{1}{2} r^\sigma |\partial_t u(x, t)| &\leq \left| \frac{1}{2} r^\sigma \partial_t u(x, t) - (u(x, t_0) - u(x, t)) \right| + 2 \|u\|_{C(Q_r)} \\ &= \frac{1}{2} r^\sigma |\partial_t u(x, t) - \partial_t u(x, t_0^x)| + 2 \|u\|_{C(Q_r)} \\ &\leq \frac{1}{2} r^{2+\alpha} [\partial_t u]_{C_t^{\frac{2+\alpha-\sigma}{\sigma}}(Q_r)} + 2 \|u\|_{C(Q_r)}. \end{aligned}$$

Hence this implies the required inequality. \square

Lemma 2.5. *Let $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$, and let $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution of the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$. If $u \in$, then we have the estimates

$$\begin{aligned} [D^2 u]_{C_t^{\frac{\alpha}{\sigma}}(Q_r)} &\lesssim \|D^2 u\|_{C(Q_r)} + \|u\|_{L_T^\infty(L_\omega^1)}, \\ [\partial_t u]_{C_x^{2+\alpha-\sigma}(Q_r)} &\lesssim \|u\|_{L_T^\infty(L_\omega^1)} \end{aligned}$$

for any $r \in (0, 1)$.

Proof. Take any $r \in (0, 2)$ and $(x, t) \in Q_r$. We note that $0 < \alpha < 2 + \alpha - \sigma < 1$. For h with $|h| < \epsilon$, we consider the difference quotients in the x -direction

$$u^h(x, t) = \frac{u(x+h, t) - u(x, t)}{|h|}.$$

Write $u^h = u_1^h + u_2^h$ where $u_1^h = u^h \mathbb{1}_{Q_r}$. By Theorem 2.4 [KL3], we have that $\mathbf{M}_{\mathfrak{L}_2}^+ u^h - \partial_t u^h \geq 0$ and $\mathbf{M}_{\mathfrak{L}_2}^- u^h - \partial_t u^h \leq 0$ on Q_r . Since $\partial_t u_2^h \equiv 0$ in Q_r , it follows from the uniform ellipticity (1.7) of $\mathbf{M}_{\mathfrak{L}_2}^+$ and $\mathbf{M}_{\mathfrak{L}_2}^-$ with respect to \mathfrak{L}_2 that

$$\mathbf{M}_{\mathfrak{L}_0}^+ u_1^h - \partial_t u_1^h \geq -\mathbf{M}_{\mathfrak{L}_2}^+ u_2^h \quad \text{and} \quad \mathbf{M}_{\mathfrak{L}_0}^- u_1^h - \partial_t u_1^h \leq -\mathbf{M}_{\mathfrak{L}_2}^- u_2^h \quad \text{in } Q_r.$$

Then it is easy to show that $|\mathbf{M}_{\mathfrak{L}_2}^+ u_2^h| \vee |\mathbf{M}_{\mathfrak{L}_2}^- u_2^h| \lesssim \|u\|_{L_T^\infty(L_\omega^1)}$ in Q_r for a universal constant $c > 0$. So we have that

$$\mathbf{M}_{\mathfrak{L}_0}^+ u_1^h - \partial_t u_1^h \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{and} \quad \mathbf{M}_{\mathfrak{L}_0}^- u_1^h - \partial_t u_1^h \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_r.$$

We now consider another difference quotients in the x -direction

$$w^h(x, t) = \frac{u_1^h(x+h, t) - u_1^h(x, t)}{|h|}.$$

Applying Theorem 2.4 [KL3] again, we obtain that

$$\mathbf{M}_{\mathfrak{L}_0}^+ w^h - \partial_t w^h \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{and} \quad \mathbf{M}_{\mathfrak{L}_0}^- w^h - \partial_t w^h \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_r.$$

From the Hölder estimate(Theorem 2.5) in [KL4], we get the estimate

$$[w^h]_{C_t^{\frac{\alpha}{\sigma}}(Q_r)} \leq [w^h]_{C^\alpha(Q_r)} \lesssim \|w^h\|_{C(Q_r)} + \|w^h\|_{L_T^\infty(L_\omega^1)} + \|u\|_{L_T^\infty(L_\omega^1)}.$$

By the mean value theorem, we easily have that $\|w^h\|_{C(Q_{r_\epsilon})} \leq \|D^2 u\|_{C(Q_{r_\epsilon})}$. Since $|D\omega(y, s)| + |D^2\omega(y, s)| \lesssim \omega(y)$, it follows from the integration by parts that $\|w^h\|_{L_T^\infty(L_\omega^1)} \leq \|u\|_{L_T^\infty(L_\omega^1)}$. Thus we obtain that

$$[w^h]_{C_t^{\frac{\alpha}{\sigma}}(Q_r)} \lesssim \|D^2 u\|_{C(Q_r)} + \|u\|_{L_T^\infty(L_\omega^1)}.$$

Taking the limit $|h| \rightarrow 0$, the first inequality can be obtained.

Take any $(x, t) \in Q_r$. Then it follows from the uniform ellipticity that

$$(2.2) \quad \begin{aligned} \mathbf{M}_2^-(\tau_x^t u - \tau^t u)(0, 0) &\leq \partial_t u(x, t) - \partial_t u(0, t) \\ &= \mathbf{I}u(x, t) - \mathbf{I}u(0, t) \leq \mathbf{M}_2^+(\tau_x^t u - \tau^t u)(0, 0) \end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function satisfying that $\varphi = 1$ in B_1 , $\varphi = 0$ in $\mathbb{R}^n \setminus B_{3/2}$ and $0 \leq \varphi \leq 1$ in \mathbb{R}^n , and take any $L \in \mathfrak{L}_2$. Then by the change of variable, the mean value theorem and (1.3) we have that

$$(2.3) \quad \begin{aligned} L(\tau_x^t u - \tau^t u)(0, 0) &= \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_t(u, 0, y)] \varphi(y) K(y) dy \\ &\quad + \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_t(u, 0, y)] (1 - \varphi(y)) K(y) dy \\ &\lesssim \varphi^+ u(x, 0) + \|u\|_{L_T^\infty(L_\omega^1)} |x|, \end{aligned}$$

where

$$\varphi^+ u(x, 0) = \sup_{t \in (-T, 0]} \sup_{K \in \mathcal{K}_2} \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_t(u, 0, y)] \varphi(y) K(y) dy.$$

Similarly we can obtain that

$$(2.4) \quad L(\tau_x^t u - \tau^t u)(0, 0) \gtrsim \varphi^- u(x, 0) - \|u\|_{L_T^\infty(L_\omega^1)} |x|,$$

where

$$\varphi^- u(x, 0) = \inf_{t \in (-T, 0]} \inf_{K \in \mathcal{K}_2} \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_t(u, 0, y)] \varphi(y) K(y) dy.$$

The estimates (2.2), (2.3) and (2.4) imply that

$$(2.5) \quad \begin{aligned} \varphi^- u(x, 0) - \|u\|_{L_T^\infty(L_\omega^1)} |x| &\lesssim \mathbf{M}_2^-(\tau_x^t u - \tau^t u)(0, 0) \\ &\leq \partial_t u(x, t) - \partial_t u(0, t) \\ &\leq \mathbf{M}_2^+(\tau_x^t u - \tau^t u)(0, 0) \\ &\lesssim \varphi^+ u(x, 0) + \|u\|_{L_T^\infty(L_\omega^1)} |x|. \end{aligned}$$

Applying the method in Lemma 9.2 [CS1] with (2.5), we have that

$$|\varphi^- u(x, 0)| \vee |\varphi^+ u(x, 0)| \lesssim \|u\|_{L_T^\infty(L_\omega^1)} |x|^\beta$$

for some $\beta \in (0, 1)$. Here, without loss of generality, we may assume that $\beta = 2 + \alpha - \sigma$ by applying a standard telescopic argument [CC]. Hence the second inequality can be achieved from a standard translation argument. Therefore we complete the proof. \square

We now consider the class \mathfrak{L}_* of operators L with kernels $K \in \mathcal{K}_*$ satisfying (1.2) such that there are some $\varrho_0 > 0$ and a constant $C > 0$ such that

$$(2.6) \quad |\nabla K(y)| \leq C \omega(y) \quad \text{for any } y \in \mathbb{R}^n \setminus B_{\varrho_0}.$$

We note that \mathfrak{L}_1 is the largest scale invariant class contained in the class \mathfrak{L}_* .

Theorem 2.6. *Let $\sigma \in [\sigma_0, 2)$ for some $\sigma_0 \in (1, 2)$. Then there is some $\varrho_0 > 0$ (depending on $\lambda, \Lambda, \sigma_0$ and n) so that if \mathbf{I} is a nonlocal, translation-invariant and uniformly elliptic operator with respect to \mathfrak{L}_* and $u \in L_T^\infty(L_\omega^1)$ satisfies the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2,$$

then there is some $\alpha > 0$ such that

$$\|Du\|_{C_t^{\frac{\alpha}{\sigma}}(Q_r)} \lesssim \|Du\|_{C(Q_r)} + \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $r \in (0, 2)$.

Proof. We proceed the proof by applying Theorem 3.4 [KL4] to the difference quotients in the x -direction

$$w^h(x, t) = \frac{u(x+h, t) - u(x, t)}{|h|}.$$

Take any $r \in (0, 2)$. Then we write $w^h = w_1^h + w_2^h$ where $w_1^h = w^h \mathbb{1}_{Q_r}$. From Theorem 2.4 [KL3], we have that $\mathbf{M}_{\mathfrak{L}^*}^+ w^h - \partial_t w^h \geq 0$ and $\mathbf{M}_{\mathfrak{L}^*}^- w^h - \partial_t w^h \leq 0$ in Q_r . Because $\partial_t w_2^h \equiv 0$ in Q_r , it follows from the uniform ellipticity with respect to \mathfrak{L}^* that we get that

$$\begin{aligned} \mathbf{M}_{\mathfrak{L}_0}^+ w_1^h - \partial_t w_1^h &\geq \mathbf{M}_{\mathfrak{L}_*}^+ w_1^h - \partial_t w_1^h \geq \mathbf{M}_{\mathfrak{L}_*}^+ w^h - \mathbf{M}_{\mathfrak{L}_*}^+ w_2^h - \partial_t w^h \geq -\mathbf{M}_{\mathfrak{L}_*}^+ w_2^h \quad \text{in } Q_r, \\ \mathbf{M}_{\mathfrak{L}_0}^- w_1^h - \partial_t w_1^h &\leq \mathbf{M}_{\mathfrak{L}_*}^- w_1^h - \partial_t w_1^h \leq \mathbf{M}_{\mathfrak{L}_*}^- w^h - \mathbf{M}_{\mathfrak{L}_*}^- w_2^h - \partial_t w^h \leq -\mathbf{M}_{\mathfrak{L}_*}^- w_2^h \quad \text{in } Q_r. \end{aligned}$$

If we can show that $|\mathbf{M}_{\mathfrak{L}_*}^+ w_2^h| \vee |\mathbf{M}_{\mathfrak{L}_*}^- w_2^h| \lesssim \|u\|_{L_T^\infty(L_\omega^1)}$ in Q_r , then we have that

$$\mathbf{M}_{\mathfrak{L}_0}^+ w_1^h - \partial_t w_1^h \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{and} \quad \mathbf{M}_{\mathfrak{L}_0}^- w_1^h - \partial_t w_1^h \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_r$$

for h with a sufficiently small $|h|$. Indeed, by using (2.6), it can be obtained from the fact that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_\rho} |u(x+y, t)| \frac{|K(x, y, t) - K(x, y-h, t)|}{|h|} dy \\ &+ \int_{\mathbb{R}^n \setminus B_\rho} |u(x+y+h, t)| K(x, y, t) dy \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \end{aligned}$$

for some $\rho > 0$. Hence w_1^h admits the Hölder estimate (Theorem 3.4 [KL4]) on Q_r , and thus applying the mean value theorem and integration by parts with (2.6) gives the estimate

$$\|w_1^h\|_{C_t^{\frac{\alpha}{\sigma}}(Q_r)} \leq \|Du\|_{C(Q_r)} + \|u\|_{L_T^\infty(L_\omega^1)}.$$

Finally, taking the limit $|h| \rightarrow 0$, we obtain the required result. \square

Remark. In order to show Theorem 1.1, we learned from the interpolation results obtained in this section that the norm $\|u\|_{C^{2,\alpha}(Q_r)}$ of viscosity solutions $u \in L_T^\infty(L_\omega^1)$ of the equation

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_r$$

is controlled by only two seminorms $[\partial_t u]_{C_t^{\frac{2+\alpha-\sigma}{\sigma}}(Q_r)}$ and $[D^2 u]_{C_x^\alpha(Q_r)}$, and so only two norms $\|u\|_{C_x^{2,\alpha}(Q_r)}$ and $\|u\|_{C_t^{1, \frac{2+\alpha-\sigma}{\sigma}}(Q_r)}$.

3. APPROXIMATION OF SOLUTIONS AND AVERAGE OF SUBSOLUTIONS

In the first part of this section, we show that any viscosity solution of (1.5) can be approximated by $C^{2,\alpha}$ -functions solving an approximate equation with the same shape as (1.5), by using a standard regularization argument. This useful result makes it possible to extend an estimate on $C^{2,\alpha}$ -solutions to the estimate on viscosity solutions by passing to the limit process.

Let Ω be a bounded domain in \mathbb{R}^n and $\Omega_\tau = \Omega \times (-\tau, 0]$ for $\tau \in (0, T)$. Then we say that a function $u : \mathbb{R}_T^n \rightarrow \mathbb{R}$ is in $C_x^{1,1}(\Omega_\tau)$, if there is a constant $C_0 > 0$ (independent of (x, t) and (y, t)) such that

$$(3.1) \quad |u(y, t) - u(x, t) - (y - x) \cdot \nabla u(x, t)| \leq C_0 |y - x|^2$$

for all $(x, t), (y, t) \in \Omega_\tau$. Here we denote by the norm $\|u\|_{C_x^{1,1}(\Omega_\tau)}$ the smallest C_0 satisfying (3.1).

The following definitions are the parabolic version corresponding to the elliptic case in [CS1] (see also [KL4]).

Definition 3.1. For a nonlocal parabolic operator \mathbf{I} and $\tau \in (0, T)$, we define $\|\mathbf{I}\|$ in Ω_τ with respect to a weight ω as

$$\|\mathbf{I}\| = \sup_{(y,s) \in \Omega_\tau} \sup_{u \in \mathcal{F}_{y,s}^M} \frac{|\mathbf{I}u(y, s)|}{1 + \|u\|_{L_T^\infty(L_\omega^1)} + \|u\|_{C_x^{1,1}(Q_1(y,s))}}$$

where $\mathcal{F}_{y,s}^M = \{u \in \mathfrak{F}(\mathbb{R}_T^n) \cap C_x^2(y, s) : \|u\|_{L_T^\infty(L_\omega^1)} \vee \|u\|_{C_x^{1,1}(Q_1(y,s))} \leq M\}$ for some $M > 0$.

For $K_\beta \in \mathfrak{L}_0$ and $\varepsilon > 0$, we consider the following regularized kernels

$$K_\beta^\varepsilon(y) = \varphi_\varepsilon(y) \frac{\lambda(2 - \sigma)}{|y|^{n+\sigma}} + (1 - \varphi_\varepsilon(y)) K_\beta(y)$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a function such that $0 \leq \varphi \leq 1$ in \mathbb{R}^n , $\varphi = 0$ in $\mathbb{R}^n \setminus B_2$ and $\varphi = 1$ in B_1 , and $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)$. Then we define the corresponding operator \mathbf{I}^ε by

$$\mathbf{I}^\varepsilon v(x, t) := \inf_\beta L_\beta^\varepsilon v(x, t) := \inf_\beta \int_{\mathbb{R}^n} \mu_t(v, x, y) K_\beta^\varepsilon(y) dy.$$

Under the parabolic topology, it is natural to consider the partial derivative ∂_t^- with respect to the past time defined by

$$\partial_t^- u(x, t) = \lim_{h \rightarrow 0^-} \frac{u(x, t+h) - u(x, t)}{h}$$

for $u \in \mathfrak{F}$, if it exists.

Lemma 3.2. Let $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution of the nonlocal parabolic concave equation

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_{1+\eta},$$

where every L_β belong to the class $\mathfrak{L}_2(\sigma)$ for $\sigma \in (1, 2)$. Then there are some $\alpha \in (0, 1)$ and a sequence $\{u^\varepsilon\} \subset C^{2,\alpha}(Q_1)$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{Q_{1+\eta}} |u^\varepsilon - u| = 0, \quad \lim_{\varepsilon \rightarrow 0} \partial_t u^\varepsilon = \partial_t u \quad \text{on } B_1 \times (-1, 0),$$

$\lim_{\varepsilon \rightarrow 0} \partial_t u^\varepsilon(x, 0) = \partial_t^- u(x, 0)$ for any $x \in B_1$ and

$$(3.2) \quad \begin{cases} \mathbf{I}^\varepsilon u^\varepsilon - \partial_t u^\varepsilon = 0 & \text{in } Q_{1+\eta}, \\ u^\varepsilon = u & \text{in } \mathbb{R}_T^n \setminus Q_{1+\eta}. \end{cases}$$

Moreover, we have that $\lim_{\varepsilon \rightarrow 0} \|\mathbf{I}^\varepsilon - \mathbf{I}\| = 0$.

Remark. Note that the condition $\lim_{\varepsilon \rightarrow 0} \|\mathbf{I}^\varepsilon - \mathbf{I}\| = 0$ implies that \mathbf{I}^ε converges weakly to \mathbf{I} in $Q_{1+\eta}$ as in [KL4].

Proof. We observe that if $L_\beta \in \mathfrak{L}_2(\sigma)$, then $L_\beta^\varepsilon \in \mathfrak{L}_2(\sigma)$. For any $\varepsilon \in (0, 1)$, let u^ε be the viscosity solution of (3.2). Then it follows from Corollary 7.9 [KL4] that $u^\varepsilon \in C^{2,\alpha}(Q_1)$ for some $\alpha \in (0, 1)$.

If $v \in \mathcal{F}_{y,s}^M$ for $M > 0$ and $(y, s) \in Q_1$, then $\|v\|_{L_T^\infty(L_\omega^1)} \vee \|v\|_{C^{1,1}(Q_1(y,s))} \leq M$ and $v \in \mathfrak{F} \cap C_x^2(y, s)$, and so we have that

$$|v(x, t) - v(y, t) - (x - y) \cdot \nabla_x v(y, t)| \leq \|v\|_{C^{1,1}(Q_1(y,s))} |x - y|^2$$

for all $(x, t) \in Q_1(y, s)$. Thus by simple computation, we obtain that

$$|\mathbf{I}^\varepsilon v(x, t) - \mathbf{I}v(x, t)| \lesssim \varepsilon^{2-\sigma},$$

so that $\|\mathbf{I}^\varepsilon - \mathbf{I}\| \lesssim \varepsilon^{2-\sigma} \rightarrow 0$ as $\varepsilon \rightarrow 0$ because $\sigma \in (0, 2)$. Thus by Lemma 5.7 [KL4] we conclude that u^ε converges to u uniformly in $Q_{1+\eta}$ as $\varepsilon \rightarrow 0$.

For $\varepsilon \in (0, 1)$, $h \in (-1, 1)$ and $(x, t) \in Q_1$, we set

$$g_{\varepsilon,h}(x, t) = \frac{u^\varepsilon(x, t+h) - u^\varepsilon(x, t)}{h} \quad \text{and} \quad g_h(x, t) = \frac{u(x, t+h) - u(x, t)}{h}.$$

For every fixed $h \in (0, 1)$, it is easy to check that $g_{\varepsilon,h}$ converges uniformly to g_h on Q_1 as $\varepsilon \rightarrow 0$, and moreover $g_{\varepsilon,h}$ has a pointwise limit $\partial_t u^\varepsilon$ on Q_1 as $h \rightarrow 0$. Thus, by commutative property of double limits, g_h has a pointwise limit on Q_1 as $h \rightarrow 0$, and moreover

$$\partial_t u(x, t) = \lim_{h \rightarrow 0} g_h(x, t) = \lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} g_{\varepsilon,h}(x, t) = \lim_{\varepsilon \rightarrow 0} \partial_t u^\varepsilon(x, t)$$

for any $(x, t) \in B_1 \times (-1, 0)$ and $\lim_{\varepsilon \rightarrow 0} \partial_t u^\varepsilon(x, 0) = \partial_t^- u(x, 0)$ for any $x \in B_1$. Hence we are done. \square

From Lemma 3.2 and Theorem 2.2 [KL3], we can easily derive the following corollary which shall be useful in the final step of the proof of the main theorem.

Corollary 3.3. *If $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution of the nonlocal parabolic concave equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_{1+\eta},$$

where every L_β belong to $\mathfrak{L}_2(\sigma)$ for $\sigma \in (1, 2)$, then $\mathbf{I}u - \partial_t u$ is well-defined on $Q_{1+\eta}$ in the classical sense and

$$\mathbf{I}u(x, t) - \partial_t u(x, t) = 0 \quad \text{for any } (x, t) \in Q_{1+\eta}.$$

In the second part, we shall show that an averages of viscosity subsolutions to the nonlocal parabolic concave equation is a viscosity subsolution to the same equation. This implies that the convolution of the viscosity subsolution with a mollifier with compact support is also a viscosity subsolution, which shall be very useful in obtaining local uniform boundedness of linear operators in Section 6.

Lemma 3.4. *If $u, v \in L_T^\infty(L_\omega^1)$ be viscosity subsolutions of the concave equations $\mathbf{I}u - \partial_t u = 0$ and $\mathbf{I}v - \partial_t v = 0$ in $\Omega \times I$, then we have that*

$$\mathbf{I}\left(\frac{u+v}{2}\right) - \partial_t\left(\frac{u+v}{2}\right) \geq 0 \quad \text{in } \Omega \times I$$

*in the viscosity sense. In particular, if $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution of the concave equation $\mathbf{I}u - \partial_t u = 0$ in Q_1 and $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a mollifier supported in a small ball B_δ such that $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$, then $\mathbf{I}(\varphi * u) - \partial_t(\varphi * u) \geq 0$ in Q_1 in the viscosity sense.*

Remark. Note that the convolution $\varphi * u$ of φ and u means

$$\varphi * u(x, t) = \int_{\mathbb{R}^n} \varphi(x - y) u(y, t) dy = \int_{\mathbb{R}^n} u(x - y, t) \varphi(y) dy, \quad x \in \mathbb{R}^n, t \in (-T, 0].$$

Proof. We consider approximate equations $\mathbf{I}^\varepsilon u^\varepsilon - \partial_t u^\varepsilon = 0$ and $\mathbf{I}^\varepsilon v^\varepsilon - \partial_t v^\varepsilon = 0$ in Q_1 with boundary values as in (3.2). By Lemma 3.2, we see that $u^\varepsilon, v^\varepsilon \in C^2(Q_1)$ and $u^\varepsilon, v^\varepsilon$ converges uniformly to u, v in Q_1 , respectively. Thus the operators $L_\beta^\varepsilon u^\varepsilon, L_\beta^\varepsilon v^\varepsilon$ are well-defined and continuous on Q_1 . Now it follows from simple computation that

$$\begin{aligned} \mathbf{I}^\varepsilon \left(\frac{u^\varepsilon + v^\varepsilon}{2} \right) - \partial_t \left(\frac{u^\varepsilon + v^\varepsilon}{2} \right) &\geq \frac{\inf_\beta L_\beta u^\varepsilon + \inf_\beta L_\beta v^\varepsilon}{2} - \partial_t \left(\frac{u^\varepsilon + v^\varepsilon}{2} \right) \\ &= \frac{(\mathbf{I}^\varepsilon u^\varepsilon - \partial_t u^\varepsilon) + (\mathbf{I}^\varepsilon v^\varepsilon - \partial_t v^\varepsilon)}{2} \geq 0 \quad \text{in } \Omega \times I \end{aligned}$$

in the viscosity sense. Since it is obvious that $\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - u\|_{L_T^\infty(L_\omega^1)} = 0$ and $\lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - v\|_{L_T^\infty(L_\omega^1)} = 0$, by Lemma 5.4 [KL4] and Lemma 3.2 we obtain the first required result. Finally, the second part is a natural by-product of the first part we obtained just before in the above. \square

4. LINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

In this section, we shall obtain regularity results for linear parabolic integro-differential equations much better than those for the nonlinear equations.

Theorem 4.1. *Let L be a linear integro-differential operator in the class $\mathfrak{L}_1(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. If $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution of*

$$Lu - \partial_t u = 0 \quad \text{in } Q_{1+\eta},$$

then $u \in C^{2,\alpha}(Q_1)$, and moreover there is some $\alpha \in (0, 1)$ such that

$$\|u\|_{C^{2,\alpha}(Q_1)} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Proof. Applying Theorem 3.6 in [KL4], we see that there is a constant $\alpha \in (0, 1)$ such that $u \in C^{1,\alpha}(Q_1)$ and

$$(4.1) \quad \|u\|_{C^{1,\alpha}(Q_1)} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

We note that $Lu_e(x, t) - \partial_t u_e(x, t) = 0$ for $(x, t) \in Q_1$ where u_e means the weak derivative of u in the direction $e \in S^{n-1}$. Also by (4.1), we note that u_e coincides with the strong type directional derivative of u in the direction e on Q_1 .

Next we show that $u_e \in L_T^\infty(L_\omega^1)$. For $(x, t) \in Q_1$, we consider a function $w \in C_0^1(\mathbb{R}^n)$ such that $w(y) = 1$ for $|y| < 1/2$, $|w_e(y)| \leq 1$ and $w(y) \geq 1$ for $1/2 \leq |y| < 1$, and $w(y) = K(y)$ for $|y| \geq 1$. Take any $(x, t) \in Q_1$. Then by integration by parts, (1.3) and Theorem 2.1, we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u_e(y, t) w(y) dy \right| &= \left| \int_{\mathbb{R}^n} u(y, t) w_e(y) dy \right| \\ (4.2) \quad &\leq \int_{1/2 \leq |y| < 1} |u(y, t)| dy + \int_{|y| \geq 1} |u(y, t)| |\langle e, \nabla K(y) \rangle| dy \\ &\lesssim \|u\|_{C(Q_{1+\eta})} + \|u\|_{L_T^\infty(L_\omega^1)} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}. \end{aligned}$$

This implies that $(u_e)^+, (u_e)^- \in L^1(w dy)$. Thus we see that $|u_e| \in L^1(w dy)$. Moreover we conclude that $u_e \in L_T^\infty(L_\omega^1)$.

Then it follows from Theorem 3.6 [KL4] that $u_e \in C_x^{1,\alpha}(Q_1)$. Thus we obtain that $u \in C_x^{2,\alpha}(Q_1)$. Here we note that we could choose some $\alpha > 0$ so that $\alpha < \sigma_0 - 1$ in Theorem 3.4 [KL4] (or Theorem 5.2 [KL3]). Since $(2 + \alpha)/\sigma > 1$ for such $\alpha > 0$, we see that

$$\frac{2 + \alpha - \sigma}{\sigma} + 1 = \frac{2 + \alpha}{\sigma}$$

and $0 < \alpha < 2 + \alpha - \sigma < 1$. Since $0 < 2 + \alpha - \sigma < 1 < 1 + \alpha$, by (4.1) we can obtain that u is $C_t^{\frac{2+\alpha-\sigma}{\sigma}}$ -Hölder continuous in Q_1 . By applying the idea of the proof of Theorem 7.8 [KL4], the $C_t^{1, \frac{2+\alpha-\sigma}{\sigma}}$ -regularity of u can be achieved on Q_1 . Therefore by the final remark in Section 2, we conclude that $u \in C^{2,\alpha}(Q_1)$. \square

Let $\mathfrak{F}(\mathbb{R}_T^n)$ denote the family of all real-valued measurable functions defined on \mathbb{R}_T^n . Then we introduce a function space $L_T^\infty(L_x^2)$ consisting of all $f \in \mathfrak{F}(\mathbb{R}_T^n)$ satisfying

$$\sup_{t \in (-T, 0]} \left(\int_{\mathbb{R}^n} |f(x, t)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Theorem 4.2. *If $\|L_0 u\|_{L_T^\infty(L_x^2)} < \infty$ for some $L_0 \in \mathfrak{L}_0(\sigma)$ with $\sigma \in (0, 2)$ and $u \in \mathfrak{F}$, then we have that*

$$\sup_{L \in \mathfrak{L}_0(\sigma)} \|Lu\|_{L_T^\infty(L_x^2)} \lesssim \inf_{L \in \mathfrak{L}_0(\sigma)} \|Lu\|_{L_T^\infty(L_x^2)}.$$

Proof. If we denote the Fourier transform of $u \in \mathfrak{F}(\mathbb{R}_T^n)$ in terms of space variable by $\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x, t) dx$, then it follows from Plancherel's Theorem that

$$\widehat{Lu}(\xi, t) = \left(- \int_{\mathbb{R}^n} 2(1 - \cos(y \cdot \xi)) K(y) dy \right) \widehat{u}(\xi, t) := -m(\xi) \widehat{u}(\xi, t)$$

for any $L \in \mathfrak{L}_0(\sigma)$. By simple computation as in [CS1], we have that

$$\frac{1}{c_0} |\xi|^\sigma \leq m(\xi) \leq c_0 |\xi|^\sigma$$

for a universal constant $c_0 > 0$ possibly depending on λ, Λ and the dimension n , but not depending on t . Applying standard harmonic analysis, there is a universal constant $C > 0$ possibly depending on λ, Λ and the dimension n , but not depending on t such that

$$\sup_{t \in (-T, 0]} \sup_{\|v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \neq 0} \frac{\|L_1 \circ L_2^{-1} v(\cdot, t)\|_{L^2(\mathbb{R}^n)}}{\|v(\cdot, t)\|_{L^2(\mathbb{R}^n)}} = \|m_1 m_2^{-1}\|_{L^\infty(\mathbb{R}^n)} < C < \infty$$

for any $L_1, L_2 \in \mathfrak{L}_0(\sigma)$, where m_1 and m_2^{-1} denote the symbols of L_1 and the inverse L_2^{-1} of the operator L_2 , respectively. Hence this implies the required result. \square

Let s be a real number. Then the homogeneous mixed Sobolev space $L_T^\infty(\dot{H}_x^s)$ is defined as the function space of all $f \in \mathfrak{F}(\mathbb{R}_T^n)$ satisfying

$$\|f\|_{L_T^\infty(\dot{H}_x^s)} := \sup_{t \in (-T, 0]} \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi, t)|^2 \right)^{\frac{1}{2}} < \infty.$$

For $p_\sigma := 2n/(n-2\sigma)$ with $\sigma \in (0, 2)$, we define a function space $L_T^\infty(L_x^{p_\sigma})$ consisting of all $f \in \mathfrak{F}(\mathbb{R}_T^n)$ satisfying

$$\|f\|_{L_T^\infty(L_x^{p_\sigma})} := \sup_{t \in (-T, 0]} \left(\int_{\mathbb{R}^n} |f(x, t)|^{p_\sigma} \right)^{\frac{1}{p_\sigma}} < \infty.$$

For $r > 0$, we consider the function space of all measurable functions f on Q_r such that

$$\|f\|_{L_t^\infty L_x^2(Q_r)} := \sup_{t \in (-r^\sigma, 0]} \left(\int_{B_r} |f(x, t)|^2 \right)^{\frac{1}{2}} < \infty.$$

Theorem 4.3. *Suppose that a function $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution of the equation*

$$L_0 u - \partial_t u = h \quad \text{in } Q_{1+\eta}$$

for $h \in L_T^\infty(L_x^2)$, where $L_0 \in \mathfrak{L}_0(\sigma)$ for $\sigma \in [\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Then there is a solution $v \in L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}})$ of the equation $L_0 v - \partial_t v = h \mathbb{1}_{Q_{1+\eta}}$ in \mathbb{R}_T^n such that

$$\sup_{L \in \mathfrak{L}_0(\sigma)} \|Lu\|_{L_t^\infty L_x^2(Q_{1/2})} \lesssim \sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)} + \|h\|_{L_T^\infty(L_x^2)} + \|v\|_{L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}})}.$$

Proof. Take any $L \in \mathfrak{L}_0(\sigma)$ for $\sigma \in [\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Let $v \in L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}})$ be a solution of the equation $L_0 v - \partial_t v = h \mathbb{1}_{Q_{1+\eta}}$ in \mathbb{R}_T^n . By the Sobolev embedding theorem, we see that

$$(4.3) \quad v \in L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}}) \subset L_T^\infty(L_x^{p_\sigma}).$$

Since $v \in L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}})$ is equivalent to $(-\Delta)^{\sigma/2} v \in L_T^\infty(L_x^2)$, it follows from Lemma 4.2 that $L_0 v \in L_T^\infty(L_x^2)$, and so $\partial_t v \in L_T^\infty(L_x^2)$. By Hölder's inequality and (4.3), we have that $v \in L_T^\infty(L_\omega^1)$. From Theorem 4.1, we obtain that

$$(4.4) \quad \|u - v\|_{C^{2,\alpha}(Q_1)} \lesssim \sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)}.$$

Since $\mu_t(u - v, x, y) = \int_0^1 \int_0^1 \langle D^2(u - v)((x + \tau y) - 2s\tau y, t)y, y \rangle ds d\tau$ by the mean value theorem, we have that

$$|\mu_t(u - v, x, y)| \lesssim \left(\sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)} \right) |y|^2$$

for any $(x, t) \in Q_{1/2}$ and $y \in B_{\frac{1}{2}+\eta}$. So we get that

$$(4.5) \quad \begin{aligned} |L(u - v)(x, t)| &\lesssim \left(\sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)} \right) \int_{|y| < \frac{1}{2}+\eta} |y|^2 K(y) dy \\ &\quad + (u - v) * K_\eta(y) \lesssim \sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)} \end{aligned}$$

for any $(x, t) \in Q_{1/2}$, where $K_\eta(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{\frac{1}{2}+\eta}}(y) K(y)$. This implies that

$$\|L(u - v)\|_{L_t^\infty L_x^2(Q_{1/2})} \lesssim \sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)}.$$

Hence we conclude that

$$\sup_{L \in \mathfrak{L}_0(\sigma)} \|Lu\|_{L_t^\infty L_x^2(Q_{1/2})} \lesssim \sup_{Q_{1+\eta}} |u - v| + \|u - v\|_{L_T^\infty(L_\omega^1)} + \|h\|_{L_T^\infty(L_x^2)} + \|v\|_{L_T^\infty(\dot{H}_x^{\frac{\sigma}{2}})}. \quad \square$$

5. LOCAL UNIFORM UPPER BOUNDEDNESS OF VISCOSITY SUBSOLUTIONS

In this section, local uniform upper boundedness of viscosity subsolutions in $L_T^\infty(L_\omega^1)$ will be achieved by using almost the same idea of the proof of the Harnack inequality in [KL3].

Theorem 5.1. *Let $\sigma \in (1, 2)$. If $u \in L_T^\infty(L_\omega^1) \cap C(Q_2)$ satisfies the equation*

$$\mathbf{M}_0^+ u - \partial_t u \geq -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_2$$

in the viscosity sense, then we have the estimate

$$\sup_{Q_{1/2}} u \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Proof. Without loss of generality, we may assume that $u \in B(\mathbb{R}_T^n)$. Indeed, if we set $u_1 = u \mathbb{1}_{Q_2}$ and $u_2 = u \mathbb{1}_{\mathbb{R}_T^n \setminus Q_2}$, then it easily follows that

$$\mathbf{M}_0^+ u_1 - \partial_t u_1 \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_2.$$

Since u is continuous on Q_2 , u_1 is bounded on \mathbb{R}_T^n . So we could use u_1 instead of u . Also we may assume that $\|u\|_{L_T^\infty(L_\omega^1)} = 1$ by dividing u by the norm $\|u\|_{L_T^\infty(L_\omega^1)}$. Thus it suffices to show that $\sup_{Q_{1/2}} u \leq C$. If u is non-positive on $Q_{1/2}$, then there is nothing to prove it. Thus we may now suppose that u is non-negative on $Q_{1/2}$. We set $s_0 = \inf\{s > 0 : u(x, t) \leq s d((x, t), \partial_p Q_1)^{-n-\sigma}, \forall (x, t) \in Q_1\}$. Then we see that $s_0 > 0$ and there is some $(\check{x}, \check{t}) \in Q_1$ such that

$$u(\check{x}, \check{t}) = s_0 d((\check{x}, \check{t}), \partial_p Q_1)^{-n-\sigma} = s_0 d_0^{-n-\sigma}$$

where $d_0 = d((\check{x}, \check{t}), \partial_p Q_1) \leq 2^{1/\sigma} < 2$ for $\sigma \in (1, 2)$. We note that

$$(5.1) \quad B_r^d(x_0, t_0) \subset Q_r(x_0, t_0) \subset B_{2r}^d(x_0, t_0)$$

for any $r > 0$ and $(x_0, t_0) \in \mathbb{R}_T^n$.

To finish the proof, we have only to show that s_0 can not be too large because $u(x, t) \leq C_1 d((x, t), \partial_p Q_1)^{-n-\sigma} \leq C$ for any $(x, t) \in Q_{1/2} \subset Q_1$ if $C_1 > 0$ is some constant with $s_0 \leq C_1$. Assume that s_0 is very large. Then by Chebyshev's inequality we have that

$$|\{u \geq u(\check{x}, \check{t})/2\} \cap Q_1| \leq \frac{2}{|u(\check{x}, \check{t})|} \|u\|_{L^\infty(L_\omega^1)} \lesssim s_0^{-1} d_0^{n+\sigma}.$$

Since $B_r^d(\check{x}, \check{t}) \subset Q_1$ and $|B_r^d| = C d_0^{n+\sigma}$ for $r = d_0/2 \leq 2^{-(1-1/\sigma)} < 1$ for $\sigma \in (1, 2)$, we easily obtain that

$$(5.2) \quad |\{u \geq u(\check{x}, \check{t})/2\} \cap B_r^d(\check{x}, \check{t})| \lesssim s_0^{-1} |B_r^d|.$$

In order to get a contradiction, we estimate $|\{u \leq u(\check{x}, \check{t})/2\} \cap B_{\delta r}^d(\check{x}, \check{t})|$ for some very small $\delta > 0$ (to be determined later). For any $(x, t) \in B_{2\delta r}^d(\check{x}, \check{t})$, we have that $u(x, t) \leq s_0 (d_0 - \delta d_0)^{-n-\sigma} = u(\check{x}, \check{t}) (1 - \delta)^{-n-\sigma}$ for $\delta > 0$ so that $(1 - \delta)^{-n-\sigma}$ is close to 1. We consider the function

$$v(x, t) = \frac{u(\check{x}, \check{t})}{(1 - \delta)^{n+\sigma}} - u(x, t).$$

Then we see that $v \geq 0$ on $B_{2\delta r}^d(\check{x}, \check{t})$, and also $\mathbf{M}_0^- v - \partial_t v \leq 1$ on $Q_{\delta r}(\check{x}, \check{t})$ because $\mathbf{M}_0^+ u - \partial_t u \geq -1$ on $Q_{\delta r}(\check{x}, \check{t})$. In order to apply Theorem 4.12 [KL3] to v , we consider $w = v^+$ instead of v . Since $w = v + v^-$, we have that

$$(5.3) \quad \mathbf{M}_0^- w - \partial_t w \leq \mathbf{M}_0^- v - \partial_t v + \mathbf{M}_0^+ v^- - \partial_t v^- \leq 1 + \mathbf{M}_0^+ v^- - \partial_t v^-$$

on $Q_{\delta r}(\check{x}, \check{t})$. Since $v^- \equiv 0$ on $B_{2\delta r}^d(\check{x}, \check{t})$, if $(x, t) \in Q_{\delta r}(\check{x}, \check{t})$ then we have that $\mu_t(v^-, x, y) = v^-(x + y, t) + v^-(x - y, t)$ for $y \in \mathbb{R}^n$.

Take any $(x, t) \in Q_{\delta r}(\check{x}, \check{t})$ and any $\varphi \in C_{Q_{\delta r}(\check{x}, \check{t})}^2(v^-; x, t)^+$. Since $(x, t) + Q_{\delta r} \subset Q_{2\delta r}(\check{x}, \check{t})$ and $v^-(x, t) = 0$, we see that $\partial_t \varphi(x, t) = 0$. Thus we have that

$$\begin{aligned} \mathbf{M}_0^+ v^-(x, t) - \partial_t \varphi(x, t) &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \mu_t^+(v^-, x, y) - \lambda \mu_t^-(v^-, x, y)}{|y|^{n+\sigma}} dy \\ &\leq 2(2 - \sigma) \Lambda \int_{\{y \in \mathbb{R}^n : v(x+y, t) < 0\}} \frac{-v(x+y, t)}{|y|^{n+\sigma}} dy \\ &\leq 2(2 - \sigma) \Lambda \int_{B_{\delta r}^c} \frac{(u(x+y, t) - (1 - \delta)^{-n-\sigma} u(\check{x}, \check{t}))_+}{|y|^{n+\sigma}} dy \\ &\leq C(2 - \sigma) \Lambda ((\delta r)^{-n-\sigma} + 1) \int_{\mathbb{R}^n} \frac{|u(y, t)|}{1 + |y|^{n+\sigma}} dy. \end{aligned}$$

This implies that

$$\mathbf{M}_0^+ v^- - \partial_t v^- \lesssim \|u\|_{L_T^\infty(L_\omega^1)} (\delta r)^{-n-\sigma} \lesssim (\delta r)^{-n-\sigma} \text{ on } Q_{\delta r}(\check{x}, \check{t}).$$

Thus by (5.3), we obtain that w satisfies

$$\mathbf{M}_0^- w(x, t) - \partial_t w \lesssim (\delta r)^{-n-\sigma} \text{ on } Q_{\delta r}(\check{x}, \check{t})$$

in viscosity sense. Since $u(\check{x}, \check{t}) = s_0 d_0^{-\beta} = 2^{-\beta} s_0 r^{-\beta}$, by Theorem 4.12 [KL3] there is some $\varepsilon_* > 0$ such that

$$\begin{aligned} |\{u \leq u(\check{x}, \check{t})/2\} \cap B_{\delta r/2}^d(\check{x}, \check{t})| &\leq |\{u \leq u(\check{x}, \check{t})/2\} \cap Q_{\delta r/2}(\check{x}, \check{t})| \\ &= |\{w \geq u(\check{x}, \check{t})((1 - \delta)^{-\beta} - 1/2)\} \cap Q_{\delta r/2}(\check{x}, \check{t})| \\ &\lesssim (\delta r)^{n+\sigma} [((1 - \delta)^{-\beta} - 1)u(\check{x}, \check{t}) + C(\delta r)^{-\sigma} (\delta r)^\sigma]^{\varepsilon_*} \\ &\quad \times [u(\check{x}, \check{t})((1 - \delta)^{-\beta} - 1/2)]^{-\varepsilon_*} \\ &\lesssim (\delta r)^{n+\sigma} \left[\left(\frac{(1 - \delta)^{-\beta} - 1}{(1 - \delta)^{-\beta} - 1/2} \right)^{\varepsilon_*} + \frac{s_0^{-\varepsilon_*} r^{n+\sigma}}{((1 - \delta)^{-\beta} - 1/2)^{\varepsilon_*}} \right] \\ &\lesssim (\delta r)^{n+\sigma} [((1 - \delta)^{-\beta} - 1)^{\varepsilon_*} + s_0^{-\varepsilon_*} r^{n+\sigma}]. \end{aligned}$$

We now choose $\delta > 0$ so small enough that $(\delta r)^{n+\sigma} ((1 - \delta)^{-\beta} - 1)^{\varepsilon_*} \lesssim |B_{\delta r/2}^d|/4$. Since δ was chosen independently of s_0 , if s_0 is large enough for such fixed δ then we get that $(\delta r)^{n+\sigma} s_0^{-\varepsilon_*} r^{n+\sigma} \lesssim |B_{\delta r/2}^d|/4$. Therefore we obtain that

$$|\{u \leq u(\check{x}, \check{t})/2\} \cap B_{\delta r/2}^d(\check{x}, \check{t})| \leq |B_{\delta r/2}^d|/2.$$

Thus we conclude that

$$\begin{aligned} |\{u \geq u(\check{x}, \check{t})/2\} \cap B_r^d(\check{x}, \check{t})| &\geq |\{u \geq u(\check{x}, \check{t})/2\} \cap B_{\delta r/2}^d(\check{x}, \check{t})| \\ &\geq |\{u > u(\check{x}, \check{t})/2\} \cap B_{\delta r/2}^d(\check{x}, \check{t})| \\ &\geq |B_{\delta r/2}^d(\check{x}, \check{t})| - |B_{\delta r/2}^d|/2 \\ &= |B_{\delta r/2}^d|/2 = C|B_r^d|, \end{aligned}$$

which contradicts (5.2) if s_0 is large enough. Hence we complete the proof. \square

6. LOCAL UNIFORM BOUNDEDNESS OF LINEAR OPERATORS

The main theme of this section is to establish local uniform boundedness of linear operators from the result obtained in Section 5, which facilitate obtaining local uniform boundedness of extremal operators to be given in the next section.

Lemma 6.1. *Let $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \text{ in } Q_2.$$

If K is a symmetric kernel satisfying $K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$, then for any radial cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ supported in B_1 and with $0 \leq \varphi \leq 1$ in \mathbb{R}^n , we have that

$$\mathbf{M}_2^+ u_\varphi - \partial_t u_\varphi \geq 0 \text{ in } Q_1$$

in the viscosity sense, where

$$u_\varphi(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) \varphi(y) dy.$$

Proof. By Lemma 5.4 [KL4] and Lemma 3.2, without loss of generality we may assume that $u \in C^2(Q_1)$. So we see that integro-differential type operators like u_φ are well-defined and continuous in Q_1 . For $\ell \in \mathbb{N}$, we set $\varphi_\ell(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{4/\ell}}(y) K(y) \varphi(y)$. Then we see that $\varphi_\ell \in L^1(\mathbb{R}^n)$ for all $\ell \in \mathbb{N}$. By Lebesgue's dominated convergence theorem, we have that

$$u_\varphi = \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^n} \mu_t(u, \cdot, y) \varphi_\ell(y) dy = 2 \left(\lim_{\ell \rightarrow \infty} u * \varphi_\ell - u \|\varphi_\ell\|_{L^1} \right).$$

Now it follows from Lemma 3.4 that

$$\mathbf{I} \left(u * \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} \right) - \partial_t \left(u * \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} \right) \geq 0 \text{ in } Q_1.$$

Also we have that $\mathbf{I}u - \partial_t u = 0$ in Q_1 . Thus by applying Theorem 2.4 [KL3], we easily obtain that

$$\begin{aligned} & \mathbf{M}_2^+ (u * \varphi_\ell - u \|\varphi_\ell\|_{L^1}) - \partial_t (u * \varphi_\ell - u \|\varphi_\ell\|_{L^1}) \\ &= \|\varphi_\ell\|_{L^1} \left[\mathbf{M}_2^+ \left(u * \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} - u \right) - \partial_t \left(u * \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} - u \right) \right] \geq 0 \text{ in } Q_1 \end{aligned}$$

for any $\ell \in \mathbb{N}$. Hence we can obtain the required result by taking limit $\ell \rightarrow \infty$. \square

Lemma 6.2. *Let $u \in L_T^\infty(L_\omega^1)$ be any viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \text{ in } Q_2.$$

Then we have the estimate

$$\mathbf{M}_2^+(Lu) - \partial_t(Lu) \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \text{ in } Q_1$$

for any $L \in \mathfrak{L}_2$.

Proof. By Lemma 5.4 [KL4] and Lemma 3.2, without loss of generality we may assume that $u \in C^2(Q_1)$. For $\ell \in \mathbb{N}$, let $\eta_\ell(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{4/\ell}}(y) K(y)$. Take any $L \in \mathfrak{L}_2$. Then as in Lemma 6.1 we have that

$$Lu = \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^n} \mu_t(u, \cdot, y) \eta_\ell(y) dy = 2 \lim_{\ell \rightarrow \infty} (u * \eta_\ell - u \|\eta_\ell\|_{L^1}).$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be any radial cut-off function supported in B_2 such that $\varphi \equiv 1$ in $B_{3/2}$ and $0 \leq \varphi \leq 1$ in \mathbb{R}^n . We set $\phi_\ell(y) = \eta_\ell(y) \varphi(y)$ and $\psi_\ell(y) = \eta_\ell(y)(1 - \varphi(y))$. By Lemma 6.1, we have that

$$(6.1) \quad \mathbf{M}_2^+ (u * \phi_\ell - u \|\phi_\ell\|_{L^1}) - \partial_t (u * \phi_\ell - u \|\phi_\ell\|_{L^1}) \geq 0 \text{ in } Q_1.$$

Also we now estimate $\mathbf{I}(u * \psi_\ell) - \partial_t(u * \psi_\ell)$ in Q_1 . Take any point $(x, t) \in Q_1$. We note that

$$\begin{aligned} \mathbf{I}(u * \psi_\ell) - \partial_t(u * \psi_\ell) &= \inf_{\beta} L_{\beta}(u * \psi_\ell) - \partial_t(u * \psi_\ell) \\ &= \inf_{\beta} u * (L_{\beta}\psi_\ell) - \partial_t(u * \psi_\ell) \end{aligned}$$

and

$$\begin{aligned} u * L_{\beta}(\psi_\ell)(x, t) &= \int_{\mathbb{R}^n} u(x - y, t) \int_{|z| \geq \frac{1}{2}} \mu(\psi_\ell, y, z) K(z) dz dy \\ &\quad + \int_{|y| \geq 1} u(x - y, t) \int_{|z| < \frac{1}{2}} \mu(\psi_\ell, y, z) K(z) dz dy \\ &:= I(x, t) + II(x, t) \end{aligned}$$

by the definition of ψ_ℓ . Then it is easy to check that

$$(6.2) \quad I = 2(u * \psi_\ell * \eta_2) - 2c(u * \psi_\ell)$$

for a universal constant $c > 0$. By the mean value theorem and triangle inequality, we see that for any $y \in \mathbb{R}^n \setminus B_1$ and $z \in B_{1/2}$,

$$\mu(\psi_\ell, y, z) = \int_0^1 \int_0^1 \langle D^2 \psi_\ell((y + \tau z) - 2s\tau z), z \rangle ds d\tau,$$

$$|(y + \tau z) - 2s\tau z| = |y + \tau(1 - 2s)z| \geq |y| - |z| \geq |y|/2.$$

Since $D^2 \psi_\ell = (D^2 \eta_\ell)(1 - \varphi) - 2(D\eta_\ell)(D\varphi) - \eta_\ell(D^2 \varphi)$, by (1.2) and (1.4) we have that

$$|D^2 \psi_\ell((y + \tau z) - 2s\tau z, t)| \leq \frac{C}{|y|^{n+\sigma}} \mathbb{1}_{\mathbb{R}^n \setminus B_3}(y) := k(y)$$

for any $y \in \mathbb{R}^n \setminus B_1$, $z \in B_{1/2}$ and $s, \tau \in [0, 1]$. Thus we obtain that

$$(6.3) \quad |II(x, t)| \leq |u| * k(x, t) \int_{|z| < \frac{1}{2}} |z|^2 K(z) dz \lesssim |u| * k(x, t) \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Hence it easily follows from (6.2), (6.3) and Young's inequality that

$$(6.4) \quad |u * L_{\beta}(\psi_\ell)(x, t)| \lesssim \|u\|_{L_T^\infty(L_\omega^1)}$$

for any β , and thus we have that

$$(6.5) \quad \mathbf{I}(u * \psi_\ell)(x, t) \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)}.$$

Since $u \in C_t^1(Q_1)$, as in the above estimate we can obtain that

$$\begin{aligned} \partial_t(u * \psi_\ell)(x, t) &= (\partial_t u) * \psi_\ell(x, t) = (\mathbf{I}u) * \psi_\ell(x, t) \\ (6.6) \quad &\leq (L_{\beta}u) * \psi_\ell(x, t) = u * (L_{\beta}\psi_\ell)(x, t) \\ &\lesssim \|u\|_{L_T^\infty(L_\omega^1)}. \end{aligned}$$

Hence by (6.1), (6.5) and (6.6), we conclude that

$$\mathbf{M}_2^+(Lu) - \partial_t(Lu) \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_1.$$

Therefore we complete the proof. \square

Lemma 6.3. *If $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \text{ in } Q_{2+\eta}$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$, then we have the estimate

$$\mathbf{L}u \lesssim \|u\|_{L_T^\infty(L_\omega^1)} \text{ in } Q_{1/4}$$

for any $\mathbf{L} \in \mathfrak{L}_2$.

Proof. By Lemma 5.4 [KL4] and Lemma 3.2, without loss of generality we may assume that $u \in C^2(Q_2)$. By Lemma 6.2, we see that

$$(6.7) \quad \mathbf{M}_0^+(\mathbf{L}u) - \partial_t(\mathbf{L}u) \geq \mathbf{M}_2^+(\mathbf{L}u) - \partial_t(\mathbf{L}u) \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \text{ in } Q_1$$

for any $\mathbf{L} \in \mathfrak{L}_2$. Since it is easy to check that \mathbf{L} is a nonlocal parabolic operator, we see that $\mathbf{L}u \in C(Q_2)$ (see [KL4]).

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in B_2 , $\varphi = 0$ in $\mathbb{R}^n \setminus B_{2+\eta/2}$ and $|D^2\varphi| \leq N_0$ in $B_{2+\eta/2}$ for some $N_0 > 0$. Then by the change of variables we have that

$$(6.8) \quad \int_{\mathbb{R}^n} \mathbf{L}u(x, t) \varphi(x) dx = \int_{\mathbb{R}^n} u(x, t) \mathbf{L}\varphi(x) dx.$$

We note that $|(x + \tau y) - 2s\tau y| = |x + \tau(1 - 2s)y| \leq |x| + |y|$ for $s, \tau \in [0, 1]$ and

$$\mu(\varphi, x, y) = \int_0^1 \int_0^1 \langle D^2\varphi((x + \tau y) - 2s\tau y), y \rangle ds d\tau$$

for any $x \in B_1$ and $y \in B_1$. We now have that

$$\mathbf{L}\varphi(x) = \int_{B_1} \mu(\varphi, x, y) K(y) dy + \int_{\mathbb{R}^n \setminus B_1} \mu(\varphi, x, y) K(y) dy := b(x) + c(x)$$

and $c(x) = 2\varphi * \eta_4(x) - 2c_0\varphi(x)$ where $c_0 = \int_{\mathbb{R}^n \setminus B_1} K(y) dy < \infty$ and $\eta_r(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_r}(y)K(y)$. Then it is easy to check that $|b(x)| \leq N_0 \int_{B_1} |y|^2 K(y) dy \leq c < \infty$ for $|x| < 5$ and $|b(x)| = 0$ for $|x| \geq 5$, and $|c(x)| \leq c$ for $|x| < 5$ and $|c(x)| \leq c/|x|^{n+\sigma}$ for $|x| \geq 5$, where $c > 0$ is a universal constant. So we see that $|\mathbf{L}\varphi(x)| \lesssim \omega(x)$. Thus by (6.8), we obtain that

$$(6.9) \quad \left| \int_{\mathbb{R}^n} \mathbf{L}u(x, t) \varphi(x) dx \right| \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

We set $\phi(x) = 1 - \varphi(x)$ and $w(x, t) = \varphi(x)\mathbf{L}u(x, t)$, and we denote by $f^x(y) = f(x+y)$. Then (6.9) implies that $w \in L_T^\infty(L_\omega^1) \cap C(Q_2)$. We now estimate $\mathbf{M}_0^+ w(x, t)$ for $x \in B_1$ and $t \in (-T, 0]$. For this, as in (6.4) we have that

$$(6.10) \quad \sup_{(x, t) \in Q_1} |u * \mathbf{L}(\phi^x K)(x, t)| \lesssim \|u\|_{L_T^\infty(L_\omega^1)},$$

because $\phi^x K$ is a smooth function with nice decay such that $\phi^x K = 0$ on $B(x; 1)$ for each $x \in B_1$. If $(x, t) \in Q_1$, then by the change of variables and (6.10), we have

the estimate

$$\begin{aligned}
(6.11) \quad L_\beta w(x, t) &= \int_{\mathbb{R}^n} \mu_t(Lu, x, y) K(y) dy - \int_{\mathbb{R}^n} \mu_t((Lu)\phi, x, y) K(y) dy \\
&= \int_{\mathbb{R}^n} \mu_t(Lu, x, y) K(y) dy - 2 \int_{\mathbb{R}^n} Lu(x + y, t) \phi^x(y) K(y) dy \\
&= \int_{\mathbb{R}^n} \mu_t(Lu, x, y) K(y) dy - 2 u * L(\phi^x K)(x, t) \\
&\gtrsim \int_{\mathbb{R}^n} \mu_t(Lu, x, y) K(y) dy - \|u\|_{L_T^\infty(L_\omega^1)}
\end{aligned}$$

for any $L_\beta \in \mathfrak{L}_2$. Hence by (6.7) and (6.11) we conclude that

$$\mathbf{M}_0^+ w - \partial_t w \gtrsim \mathbf{M}_2^+(Lu) - \partial_t(Lu) - \|u\|_{L_T^\infty(L_\omega^1)} \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{on } Q_1.$$

Therefore the required result can be achieved by applying Theorem 5.1. \square

7. LOCAL UNIFORM BOUNDEDNESS OF EXTREMAL OPERATORS

In this section, we show that if $u \in L_T^\infty(L_\omega^1)$ is a viscosity solution of the nonlocal parabolic concave equation $\mathbf{I}u - \partial_t u = 0$ in Q_2 , then $\mathbf{M}_0^+ u$ and $\mathbf{M}_0^- u$ are bounded uniformly on $Q_{1/2}$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. This plays an important role as a cornerstone in proving the main theorem in the final section.

Lemma 7.1. *Let $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2.$$

If K is a symmetric kernel with $K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$, then for any function $\gamma \in C_c^\infty(-T, T]$ with $\gamma = 1$ in $(-2^{1-\sigma}, 0]$ and $\text{supp}(\gamma) \subset (-1, \eta]$ and any radial cut-off function $\psi \in C_c^\infty(\mathbb{R}^n)$ supported in B_2 such that $\psi = 1$ in $B_{8/5}$, $\psi = 0$ in $\mathbb{R}^n \setminus B_2$ and $0 \leq \psi \leq 1$ in \mathbb{R}^n , we have that

$$\mathbf{M}_2^+(\psi\gamma u_\varphi) - \partial_t(\psi\gamma u_\varphi) \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_{1/2}$$

in the viscosity sense, where

$$u_\varphi(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) \varphi(y) dy$$

for a radial cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ supported in $B_{1/4}$ with $0 \leq \varphi \leq 1$ in \mathbb{R}^n .

Proof. By Lemma 6.1, we see that $\mathbf{M}_2^+ u_\varphi - \partial_t u_\varphi \geq 0$ in Q_1 in the viscosity sense. Set $\phi = 1 - \psi\gamma$ in \mathbb{R}_T^n . Take any $L_\beta \in \mathfrak{L}_2$ and $(x, t) \in Q_{1/2}$. Then we have that

$$(7.1) \quad L_\beta(\psi\gamma u_\varphi)(x, t) = \int_{\mathbb{R}^n} \mu_t(u_\varphi, x, y) K_\beta(y) dy - E(x, t)$$

where $E(x, t) = \int_{\mathbb{R}^n} \mu_t(\phi u_\varphi, x, y) K_\beta(y) dy$. By the mean value theorem and triangle inequality, we see that

$$(7.2) \quad \mu(\phi^x K_\beta, y, z) = \int_0^1 \int_0^1 \langle D^2(\phi^x K_\beta)((y + \tau z) - 2s\tau z), z \rangle ds d\tau$$

and $|x + (y + \tau z) - 2s\tau z| = |x + y + \tau(1 - 2s)z| \geq |y| - \frac{15}{16}|y| \geq \frac{1}{16}|y|$ for any $y \in \mathbb{R}^n \setminus B_{4/5}$, $z \in B_{1/4}$ and $-2^{1-\sigma} < t \leq 0$. Also we note that $\mu(\phi^x K_\beta, y, z) = 0$

for any $y \in B_{4/5}$, $z \in B_{1/4}$ and $-2^{1-\sigma} < t \leq 0$. Thus by (7.2) we obtain that

$$\begin{aligned}
E(x, t) &= 2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mu_t(u, x+y, z) K(z) \varphi(z) dz \right) \phi(x+y) K_\beta(y) dy \\
&= 2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mu_t(u, x+y, z) \phi(x+y) K_\beta(y) dy \right) K(z) \varphi(z) dz \\
&= 2 \int_{\mathbb{R}^n} u(x+y, t) \left(\int_{\mathbb{R}^n} \mu_t(\phi^x K_\beta, y, z) K(z) \varphi(z) dz \right) dy \\
&= 2 \int_{\mathbb{R}^n} u(x+y, t) \left(\int_{|z| < \frac{1}{4}} \mu_t(\phi^x K_\beta, y, z) K(z) \varphi(z) dz \right) dy \\
&\lesssim \int_{|y| \geq \frac{4}{5}} |u(x+y, t)| \frac{1}{|y|^{n+2+\sigma}} dy \int_{|z| < \frac{1}{4}} |z|^2 K(z) dz \lesssim \|u\|_{L_T^\infty(L_\omega^1)}
\end{aligned}$$

for any $|x| < 1/2$ and $-2^{1-\sigma} < t \leq 0$. Hence by (7.1) we conclude that

$$\begin{aligned}
&\mathbf{M}_2^+(\psi \gamma u_\varphi)(x, t) - \partial_t(\psi \gamma u_\varphi)(x, t) \\
&\geq \mathbf{M}_2^+ u_\varphi(x, t) - \partial_t u_\varphi(x, t) - E(x, t) \gtrsim -\|u\|_{L_T^\infty(\omega)}
\end{aligned}$$

for any $(x, t) \in Q_{1/2}$. Therefore we complete the proof. \square

Lemma 7.2. *Let $u \in L_T^\infty(L_\omega^1)$ be any viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Then for any operator L with a symmetric kernel K satisfying $K(y) \leq (2-\sigma)\Lambda|y|^{-n-\sigma}$, we have the estimate

$$\sup_{Q_{1/2}} |Lu| \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Proof. Take any $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. As in Lemma 6.3, without loss of generality, we may assume that $u \in C^2(Q_1)$. For convenience, we normalize $\|u\|_{L_T^\infty(L_\omega^1)} = 1$. By Lemma 6.3, we see that $\sup_{L_\beta \in \mathfrak{L}_2} |L_\beta u|$ is bounded in $Q_{1/2}$ because $-u$ is another viscosity solution of our equation. So this implies that $|\partial_t u| = |\mathbf{I}u|$ is bounded in $Q_{1/2}$. Thus it follows from that

$$\|L_\beta u - \partial_t u\|_{L_t^\infty L_x^2(Q_{1/2})} \leq \|L_\beta u\|_{L_t^\infty L_x^2(Q_{1/2})} + \|\partial_t u\|_{L_t^\infty L_x^2(Q_{1/2})} < \infty.$$

Combining Theorem 4.3 with this yields that

$$(7.3) \quad \sup_{L \in \mathfrak{L}_0(\sigma)} \|Lu\|_{L_t^\infty L_x^2(Q_{1/2})} < \infty.$$

Take any operator L with a symmetric kernel K satisfying $K(y) \leq (2-\sigma)\Lambda|y|^{-n-\sigma}$. Then we split Lu into two integrals

$$\begin{aligned}
Lu(x, t) &= \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) \varphi(y) dy + \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) (1 - \varphi(y)) dy \\
&:= u_\varphi(x, t) + u_{1-\varphi}(x, t),
\end{aligned}$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a radial cut-off function supported in B_1 such that $\varphi = 1$ in $B_{1/2}$ and $0 \leq \varphi \leq 1$ in \mathbb{R}^n . Since $K \in L^1(\mathbb{R}^n \setminus B_{1/2})$, it is easy to check that $\sup_{Q_{1/2}} |u_{1-\varphi}| < \infty$, and thus we have that

$$\|u_{1-\varphi}\|_{L_t^\infty L_x^2(Q_{1/2})} < \infty.$$

Thus by (7.3), we obtain that

$$(7.4) \quad \|u_\varphi\|_{L_t^\infty L_x^2(Q_{1/2})} < \infty.$$

From Lemma 6.1, we have that

$$(7.5) \quad \mathbf{M}_2^+ u_\varphi - \partial_t u_\varphi \geq 0 \text{ in } Q_1.$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $\psi = 1$ in $B_{1/2}$ and $\text{supp}(\psi) \subset B_{\frac{1}{2}+\eta}$, and let $\gamma \in C_c(-T, T]$ be a function such that $\gamma = 1$ in $(-2^{-\sigma}, 0]$ and $\text{supp}(\gamma) \subset (-2^{-\sigma} - \eta, \eta]$. Set $v_\varphi(x, t) = \psi(x)\gamma(t)u_\varphi(x, t)$. Then by (7.4) it is easy to check that $v_\varphi \in L_T^\infty(L_\omega^1)$. So it follows from Lemma 7.1 that

$$\mathbf{M}_2^+ v_\varphi - \partial_t v_\varphi \gtrsim -1 \text{ in } Q_{1/2}.$$

Applying Theorem 5.1, we obtain that $v_\varphi \lesssim 1$ in $Q_{1/8}$. Thus the required upper bound for Lu on $Q_{1/2}$ follows from a standard covering and scaling argument.

For the lower bound for Lu on $Q_{1/2}$, we take an operator $L_\beta \in \mathfrak{L}_2(\sigma)$ with kernel K_β and consider an operator L_* with kernel $K_* = \frac{2}{\lambda}K_\beta - \frac{1}{\lambda}K$. Then it is easy to check that

$$\frac{2-\sigma}{|y|^{n+\sigma}} \leq K_*(y) \leq \frac{(2-\sigma)(\frac{2\lambda}{\lambda} - \frac{\lambda}{\lambda})}{|y|^{n+\sigma}}.$$

As in the first half, we obtain that $L_*u \lesssim 1$ in $Q_{1/2}$. This implies that $Lu \gtrsim -1$ in $Q_{1/2}$. Therefore the required result can be achieved. \square

From the above result, it is natural to obtain the following corollaries.

Corollary 7.3. *Let $u \in L_T^\infty(L_\omega^1)$ be any viscosity solution satisfying the equation $\mathbf{I}u - \partial_t u = 0$ in Q_2 , where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Then $\mathbf{M}_0^+ u$, $\mathbf{M}_0^- u$ and $\partial_t u$ are uniformly bounded in $Q_{1/2}$, and moreover we have*

$$\left(\sup_{Q_{1/2}} |\mathbf{M}_0^+ u| \right) \vee \left(\sup_{Q_{1/2}} |\mathbf{M}_0^- u| \right) \vee \left(\sup_{Q_{1/2}} |\partial_t u| \right) \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Corollary 7.4. *Let $u \in L_T^\infty(L_\omega^1)$ be any viscosity solution satisfying the equation $\mathbf{I}u - \partial_t u = 0$ in Q_2 , where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Then we have that*

$$\sup_{Q_{1/2}} \int_{\mathbb{R}^n} |\mu_*(u, \cdot, y)| \frac{2-\sigma}{|y|^{n+\sigma}} dy \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

8. PROOF OF THE MAIN THEOREM

Let $u \in L_T^\infty(L_\omega^1)$ be any viscosity solution satisfying the equation

$$(8.1) \quad \mathbf{I}u - \partial_t u = 0 \text{ in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. From Corollary 7.4, there is a universal constant $c_0 > 0$ such that

$$(8.2) \quad \sup_{Q_{1/2}} \int_{\mathbb{R}^n} |\mu_*(u, \cdot, y)| \frac{2-\sigma}{|y|^{n+\sigma}} \varphi(y) dy \leq c_0 \|u\|_{L_T^\infty(L_\omega^1)},$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a function such that $\varphi = 1$ in B_1 , $\varphi = 0$ in $\mathbb{R}^n \setminus B_{3/2}$ and $0 \leq \varphi \leq 1$ in \mathbb{R}^n .

In order to prove Theorem 1.1, our main goal is to obtain that there is some $\alpha \in (0, 1)$ such that

$$(8.3) \quad \int_{\mathbb{R}^n} |\mu_t(u, x, y) - \mu_0(u, 0, y)| \frac{2-\sigma}{|y|^{n+\sigma}} \varphi(y) dy \lesssim (|x| + |t|^\sigma)^{\frac{\alpha}{\sigma}} \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $(x, t) \in Q_{1/2}$. This implies that the fractional Laplacian $(-\Delta)^{\sigma/2}$ admits the Hölder continuity, and moreover the viscosity solutions of the nonlocal parabolic equation in Theorem 1.1 enjoy the $C^{\sigma+\alpha}$ -regularity.

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $\psi = 1$ in $B_{1/2}$ and $\text{supp}(\psi) \subset B_{\frac{1}{2}+\eta}$, and let $\gamma \in C_c(-T, T]$ be a function such that $\gamma = 1$ in $(-2^{-\sigma}, 0]$ and $\text{supp}(\gamma) \subset (-2^{-\sigma} - \eta, \eta]$. Set $w_\varphi(x, t) = \psi(x)\gamma(t)v_\varphi(x, t)$, where

$$v_\varphi(x, t) = \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \frac{2-\sigma}{|y|^{n+\sigma}} \varphi(y) dy$$

for a radial cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ supported in $B_{1/4}$ with $0 \leq \varphi \leq 1$ in \mathbb{R}^n . Then, as in Lemma 7.2, it is easy to check that $w_\varphi \in L_T^\infty(L_\omega^1)$ and it follows from Lemma 7.1 that

$$\mathbf{M}_2^+ w_\varphi - \partial_t w_\varphi \gtrsim -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_{1/2},$$

We set

$$v_\varphi^\pm(x, t) = \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)]^\pm \frac{2-\sigma}{|y|^{n+\sigma}} \varphi(y) dy$$

and set

$$w_\varphi^S(x, t) = \psi(x)\gamma(t) \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \frac{2-\sigma}{|y|^{n+\sigma}} \varphi(y) \mathbb{1}_S(y) dy$$

for a symmetric set $S \subset \mathbb{R}^n$ (i.e. $S = -S$). Also we consider the positive part Pu and negative part Nu of w_φ defined by $Pu(x, t) = \psi(x)\gamma(t)v_\varphi^+(x, t)$ and $Nu(x, t) = \psi(x)\gamma(t)v_\varphi^-(x, t)$. Then we see that $Pu = \sup_S w_\varphi^S$ and $Nu = -\inf_S w_\varphi^S$, and moreover $Pu = w_\varphi^{S_0}$ and $Nu = -w_\varphi^{S_0^c}$ where S_0 is the symmetric set given by $S_0 = \{y \in \mathbb{R}^n : \mu_t(u, x, y) > \mu_0(u, 0, y)\}$.

Lemma 8.1. *If $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$, then there exists some $\alpha \in (0, 1)$ such that

$$Pu(x, t) \lesssim (|x|^\sigma + |t|)^{\frac{\alpha}{\sigma}} \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $(x, t) \in Q_{1/8}$.

Proof. We may assume that $\|u\|_{L_T^\infty(L_\omega^1)} \leq 1$ by dividing the equation by $\|u\|_{L_T^\infty(L_\omega^1)}$. Take any $(x, t) \in Q_{1/8}$ and $L \in \mathfrak{L}_2(\sigma)$. Then we have that

$$\begin{aligned} \mathbf{L}(\tau_x^t u - u)(0, 0) &= \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \varphi(y) K(y) dy \\ &+ \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \phi(y) K(y) dy \\ &:= \mathbf{L}_\varphi u(x, t) + \mathbf{L}_\phi u(x, t), \end{aligned} \tag{8.4}$$

where $\phi = 1 - \varphi$. Then we see that

$$w_\varphi^-(x, t) \leq \mathbf{L}_\varphi u(x, t) \leq w_\varphi^+(x, t) \tag{8.5}$$

where $w_\varphi^-(x, t) = \lambda Pu(x, t) - \Lambda Nu(x, t)$ and $w_\varphi^+(x, t) = \Lambda Pu(x, t) - \lambda Nu(x, t)$. By easy calculation, the second term in the right hand side of (8.4) becomes

$$\begin{aligned} L_\phi u(x, t) &= 2 \int_{\mathbb{R}^n} u(y, t) [K(y - x)\phi(y - x) - K(y)\phi(y)] dy \\ &\quad + C_K^\phi(t) + 2[u(0, 0) - u(x, t)] \int_{\mathbb{R}^n} K(y)\phi(y) dy \end{aligned}$$

where $C_K^\phi(t) = 2 \int_{\mathbb{R}^n} [u(y, t) - u(y, 0)]K(y)\phi(y) dy$. Thus it follows from (1.3) and Theorem 3.4 [KL4] that

$$\begin{aligned} (8.6) \quad A(x, t) &\leq \inf_{K \in \mathcal{K}_2} [L(\tau_x^t u - u)(0, 0) - C_K^\phi(t)] \\ &\leq \sup_{K \in \mathcal{K}_2} [L(\tau_x^t u - u)(0, 0) - C_K^\phi(t)] \leq B(x, t) \end{aligned}$$

for some universal constants $c, \beta > 0$, where $A(x, t) = w_\varphi^-(x, t) - c(|x|^\sigma + |t|)^{\beta/\sigma}$ and $B(x, t) = w_\varphi^+(x, t) + c(|x|^\sigma + |t|)^{\beta/\sigma}$. Here we note that β could be chosen freely in the open interval $(0, 1)$ (see [KL3]). Then we have only three possible cases; either (a) $A(x, t) \leq 0$ and $B(x, t) \geq 0$, or (b) $A(x, t) \geq 0$ and $B(x, t) \geq 0$, or (c) $A(x, t) \leq 0$ and $B(x, t) \leq 0$.

(**Case I :** (a) $A(x, t) \leq 0$ and $B(x, t) \geq 0$) (a) implies that

$$(8.7) \quad \frac{\lambda}{\Lambda} Nu(x, t) - c_1 (|x|^\sigma + |t|)^{\frac{\beta}{\sigma}} \leq Pu(x, t) \leq \frac{\Lambda}{\lambda} Nu(x, t) + c_1 (|x|^\sigma + |t|)^{\frac{\beta}{\sigma}}$$

for any $(x, t) \in Q_{1/8}$, where $c_1 = c/\Lambda$.

(**Case II :** (b) $A(x, t) \geq 0$ and $B(x, t) \geq 0$) (b) implies that

$$(8.8) \quad Nu(x, t) \leq Pu(x, t).$$

(**Case III :** (c) $A(x, t) \leq 0$ and $B(x, t) \leq 0$) (c) implies that

$$Nu(x, t) \geq Pu(x, t).$$

We note that $-u$ is another viscosity solution of (8.1). Using $-u$ instead of u , we see that $N(-u)(x, t) = Pu(x, t)$ and $P(-u)(x, t) = Nu(x, t)$. In this case, the proof can be achieved exactly in the same way as Case II. Thus we have only to consider Case I and Case II.

Our main goal is to show that there is a universal constant $c > 0$ such that $\sup_{Q_r} Pu \leq cr^\alpha$ for any small enough $r > 0$. Since $B_r^d \subset Q_r \subset B_{2r}^d$, it suffices to show that $\sup_{B_r^d} Pu \leq cr^\alpha$ for any small enough $r > 0$. If we take a rescaled function $\bar{w}_\varphi^S(x, t) = \frac{1}{c_0} w_\varphi^S(rx, r^\sigma t)$ where c_0 is the constant in (8.2), then we may assume that

(i) $|w_\varphi^S| \leq 1$ in \mathbb{R}_T^n and $\mathbf{M}_2^+ w_\varphi^S - \partial_t w_\varphi^S \geq -r^\sigma/c_0$ in B_1^d , for all symmetric sets $S \subset \mathbb{R}^n$, and

(ii) for any $(x, t) \in B_1^d$, we have that either

$$(8.9) \quad \frac{\lambda}{\Lambda} Nu(x, t) - c_1 r^\sigma (|x|^\sigma + |t|)^{\frac{\beta}{\sigma}} \leq Pu(x, t) \leq \frac{\Lambda}{\lambda} Nu(x, t) + c_1 r^\sigma (|x|^\sigma + |t|)^{\frac{\beta}{\sigma}}$$

or (8.8) holds, for any small enough $r > 0$, where c_1 is the constant in (8.7). From Lemma 3.2, we can also assume that u is C^{2, α_0} for some $\alpha_0 \in (0, 1)$, and so w_φ^S , Pu and Nu are continuous.

For our aim, we need only to prove that there are some $r \in (0, 1)$ and $\varrho \in (0, 1)$ such that

$$(8.10) \quad \sup_{B_{r^k}^d} |Pu| \leq (1 - \varrho)^k = r^{\alpha k} \quad \text{for } \alpha = \frac{\ln(1 - \varrho)}{\ln r}.$$

We are going to proceed this proof by using mathematical induction. If $k = 0$, then it is trivial by (i). Assume that (8.10) holds in the k^{th} -step ($k \in \mathbb{N}$). Then we shall show that (8.10) holds also for the $(k + 1)^{th}$ -step. By (8.10) and geometric observation, we have that

$$(8.11) \quad -1 \leq w_\varphi^S(x, t) \leq Pu(x, t) \leq \frac{1}{1 - \varrho} (|x|^\sigma + |t|)^\frac{\alpha}{\sigma}$$

for any (x, t) with $(|x|^\sigma + |t|)^{1/\sigma} > r^k$.

We consider the following rescaled functions

$$\begin{aligned} \tilde{w}_\varphi^S(x, t) &:= (1 - \varrho)^{-k} w_\varphi^S(r^k x, r^{k\sigma} t), \\ \tilde{P}u(x, t) &:= (1 - \varrho)^{-k} Pu(r^k x, r^{k\sigma} t) = \sup_S \tilde{w}_\varphi^S(x, t), \\ \tilde{N}u(x, t) &:= (1 - \varrho)^{-k} Nu(r^k x, r^{k\sigma} t) = -\inf_S \tilde{w}_\varphi^S(x, t). \end{aligned}$$

Then the function $\tilde{P}u$ satisfies that

$$\tilde{P}u(x, t) \leq \begin{cases} 1 & \text{in } B_1^d, \\ \frac{1}{1 - \varrho} (|x|^\sigma + |t|)^\frac{\alpha}{\sigma} & \text{outside } B_1^d. \end{cases}$$

Choosing $\beta = \alpha$ in (8.9), by (8.8) and (8.9) we have that

$$(8.12) \quad \frac{\lambda}{\Lambda} \tilde{N}u(x, t) - c_1 r^\sigma \leq \tilde{P}u(x, t) \leq \frac{\Lambda}{\lambda} \tilde{N}u(x, t) + c_1 r^\sigma \quad \text{in } B_1^d$$

and

$$(8.13) \quad \tilde{N}u(x, t) \leq \tilde{P}u(x, t) \quad \text{in } B_1^d.$$

Next, we shall show that if ϱ and r are chosen so small enough that $1 - \varrho = r^\alpha$ for some $\alpha \in (0, 1)$, then $\tilde{P}u \leq 1 - \varrho$ in B_r^d . This makes it possible to complete the induction process. For this proof, we assume that there are some small enough r and ϱ such that $\tilde{P}u \not\leq 1 - \varrho$ in B_r^d , i.e. $\tilde{P}u(x_0, t_0) > 1 - \varrho$ for some $(x_0, t_0) \in B_r^d$. Without loss of generality, we may suppose that (x_0, t_0) be the point at which the maximum value of $\tilde{P}u$ is attained in B_r^d . Then we see that

$$(8.14) \quad \tilde{P}u(x_0, t_0) = \tilde{w}_\varphi^{S_0}(x_0, t_0) > 1 - \varrho$$

and

$$(8.15) \quad \tilde{P}u(x, t) = \tilde{w}_\varphi^{S_0}(x, t) \leq \begin{cases} 1 & \text{in } B_1^d, \\ \frac{1}{1 - \varrho} (|x|^\sigma + |t|)^\frac{\alpha}{\sigma} & \text{outside } B_1^d, \end{cases}$$

where S_0 is the symmetric set given by $S_0 = \{y \in \mathbb{R}^n : \mu_t(u, x, y) > \mu_0(u, 0, y)\}$. Then we note that

$$(8.16) \quad \mathbf{M}_2^+ \tilde{w}_\varphi^{S_0} - \partial_t \tilde{w}_\varphi^{S_0} \geq -\frac{r^\sigma}{c_0} \left(\frac{r^\sigma}{1 - \varrho} \right)^k > -\frac{r^\sigma}{c_0} \quad \text{in } B_{1/2}^d,$$

because $\alpha < \sigma_0 < \sigma < 2$. Since it is easy to check that

$$(1 - \tilde{w}_\varphi^{S_0})_- \leq \left(\frac{1}{1 - \varrho} (|x|^\sigma + |t|)^{\frac{\alpha}{\sigma}} - 1 \right)_+ := h(x, t) \quad \text{in } \mathbb{R}_T^n$$

by (8.15), we derive that

$$(8.17) \quad \mathbf{M}_2^+(1 - \tilde{w}_\varphi^{S_0})_- \leq \mathbf{M}_2^+ h \leq c < \infty \quad \text{in } B_{1/2}^d$$

for some universal constant $c > 0$. We also observe that

$$\partial_t(1 - \tilde{w}_\varphi^{S_0})_- = 0 \quad \text{in } B_{1/2}^d,$$

because $B_1^d \subset \{(1 - \tilde{w}_\varphi^{S_0})_- = 0\}$ by (8.15). Let $v_\varphi^{S_0} = (1 - \tilde{w}_\varphi^{S_0})_+$. Then we have that $v_\varphi^{S_0}(x_0, t_0) = \inf_{B_r^d} v_\varphi^{S_0} \leq \varrho$ by (8.14), and moreover by (8.16) and (8.17) we conclude that

$$\begin{aligned} \mathbf{M}_2^- v_\varphi^{S_0} - \partial_t v_\varphi^{S_0} &\leq \mathbf{M}_2^-(1 - \tilde{w}_\varphi^{S_0}) - \partial_t(1 - \tilde{w}_\varphi^{S_0}) \\ &\quad + \mathbf{M}_2^+(1 - \tilde{w}_\varphi^{S_0})_- - \partial_t(1 - \tilde{w}_\varphi^{S_0})_- \\ &\leq -(\mathbf{M}_2^+ \tilde{w}_\varphi^{S_0} - \partial_t \tilde{w}_\varphi^{S_0}) \\ &\quad + \mathbf{M}_2^+(1 - \tilde{w}_\varphi^{S_0})_- - \partial_t(1 - \tilde{w}_\varphi^{S_0})_- \leq c \quad \text{in } B_{1/2}^d. \end{aligned}$$

By Theorem 4.11 [KL3], there are some universal constants $c > 0$ and $\mu > 0$ such that

$$(8.18) \quad |\{v_\varphi^{S_0} > \lambda \varrho\} \cap Q_r(x_0, t_0)| \leq c r^{n+\sigma} (v_\varphi^{S_0}(x_0, t_0) + c r^\sigma)^\mu (\lambda \varrho)^{-\mu}$$

for any $\lambda > 0$ and $r \in (0, 1/4)$. If we choose r so that $c r^\sigma < \varrho$, then (8.18) becomes

$$(8.19) \quad |\{v_\varphi^{S_0} > \lambda \varrho\} \cap Q_r(x_0, t_0)| \leq c r^{n+\sigma} \lambda^{-\mu} = c \lambda^{-\mu} |Q_r|$$

for any $\lambda > 0$. Set $D = \{v_\varphi^{S_0} \leq \lambda \varrho\} \cap Q_r(x_0, t_0)$. By (8.19), we have that

$$(8.20) \quad |D| \geq (1 - c \lambda^{-\mu}) |Q_r|$$

for all large enough $\lambda > 0$. Since $v_\varphi^{S_0} > \lambda \varrho \Leftrightarrow \tilde{w}_\varphi^{S_0} < 1 - \lambda \varrho$, we see that $D = \{\tilde{w}_\varphi^{S_0} \geq 1 - \lambda \varrho\} \cap Q_r(x_0, t_0)$. Since $D \subset B_1^d$ and $\tilde{P}u \leq 1$ in D by (8.15), we also see that $\tilde{P}u - \tilde{w}_\varphi^{S_0} \leq \lambda \varrho$ in D . So we have the estimate

$$(8.21) \quad \tilde{N}u + \tilde{w}_\varphi^{S_0^c} = \tilde{P}u - \tilde{w}_\varphi^{S_0} \leq \lambda \varrho \quad \text{in } D,$$

because $\tilde{w}_\varphi^{S_0} + \tilde{w}_\varphi^{S_0^c} = \tilde{P}u - \tilde{N}u$. For (Case I), it follows from (8.12) and (8.21) that

$$(8.22) \quad \tilde{w}_\varphi^{S_0^c} \leq -\frac{\lambda}{\Lambda}(1 - \lambda \varrho) + \lambda \varrho + c_1 r^\sigma \leq -\frac{\lambda}{2\Lambda} \quad \text{in } D,$$

provided that r and ϱ are chosen small enough. For (Case II), by (8.13) and (8.21) we have that

$$(8.23) \quad \tilde{w}_\varphi^{S_0^c} \leq -(1 - \lambda \varrho) + \lambda \varrho \leq -\frac{\lambda}{2\Lambda} \quad \text{in } D,$$

if r and ϱ are chosen small enough. From (8.22), (8.23) and (8.20), we obtain that

$$(8.24) \quad |\{\tilde{w}_\varphi^{S_0^c} \leq -\frac{\lambda}{2\Lambda}\} \cap Q_r(x_0, t_0)| \geq (1 - c \lambda^{-\mu}) |Q_r|.$$

for any $\lambda > 0$ and $r \in (0, 1/4)$.

For any small $\eta > 0$, let $g(x, t) = (\tilde{w}_\varphi^{S_0^c}(r\eta(x - x_0), (r\eta)^\sigma(t - t_0)) + \frac{\lambda}{2\Lambda})_+$. Then it follows from (8.24) that

$$(8.25) \quad |\{g > 0\} \cap Q_{\eta^{-1}}| \leq c\lambda^{-\mu}|Q_{\eta^{-1}}|.$$

When r is small enough, by (i) it is also easy to check that

$$(8.26) \quad \mathbf{M}_0^+ g - \partial_t g \geq -\|u\|_{L_T^\infty(L_\omega^1)} \quad \text{in } Q_2.$$

Applying Theorem 5.1 to g with small enough $r \in (0, 1/4)$, by (8.11), (8.15) and (8.25) we obtain that

$$\begin{aligned} g(x_0, t_0) &\leq C \sup_{s \in (-T, 0]} \int_{\mathbb{R}^n} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy \\ &\leq C \sup_{s \in (-T, 0]} \int_{B_{\eta^{-1}}} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy + C \sup_{s \in (-T, 0]} \int_{\mathbb{R}^n \setminus B_{\eta^{-1}}} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy \\ &\leq C \eta^{-n-\sigma} \lambda^{-\mu} + C \eta^\alpha \sup_{s \in (-T, 0]} \int_{\mathbb{R}^n \setminus B_{\eta^{-1}}} \frac{|y|^\alpha + |s|^{\alpha/\sigma}}{1 + |y|^{n+\sigma}} dy \\ &\leq C \eta^{-n-\sigma} \lambda^{-\mu} + \frac{C}{\sigma - \alpha} \eta^\sigma + \frac{C}{\sigma} \eta^{\sigma+\alpha}. \end{aligned}$$

In this estimate, choose η so small that $\frac{C}{\sigma - \alpha} \eta^\sigma + \frac{C}{\sigma} \eta^{\sigma+\alpha} < \frac{\lambda}{8\Lambda}$, and then select μ so large that $C \eta^{-n-\sigma} \lambda^{-\mu} < \frac{\lambda}{8\Lambda}$. Then we have that

$$g(x_0, t_0) \leq \frac{\lambda}{4\Lambda}.$$

This implies that $\tilde{w}_\varphi^{S_0^c}(0, 0) \leq -\frac{\lambda}{4\Lambda}$, which contradicts to the fact that $\tilde{w}_\varphi^{S_0^c}(0, 0) = 0$. Hence we conclude that $\tilde{P}u \leq 1 - \varrho$ in B_r^d , that is to say, $Pu \leq (1 - \varrho)^{k+1}$ in $B_{r^{k+1}}^d$. Therefore we complete the proof. \square

We can also obtain the following corollary in the same manner as Lemma 8.1.

Corollary 8.2. *If $u \in L_T^\infty(L_\omega^1)$ be a viscosity solution satisfying the equation*

$$\mathbf{I}u - \partial_t u = 0 \quad \text{in } Q_2,$$

where \mathbf{I} is defined on $\mathfrak{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$, then there exists some $\alpha \in (0, 1)$ such that

$$Nu(x, t) \lesssim (|x|^\sigma + |t|)^{\frac{\alpha}{\sigma}} \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $(x, t) \in Q_{1/8}$.

Proof of Theorem 1.1. As mentioned above, the case $\sigma \in (0, 1]$ could be treated in [KL4]. Thus we have only to prove our main theorem only for the case $\sigma \in (1, 2)$.

We note that the fractional Laplacian of order $\sigma \in (0, 2)$ is given by

$$-(-\Delta^{\sigma/2})u(x, t) = c_\sigma \int_{\mathbb{R}^n} \mu_t(u, x, y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy,$$

where c_σ is the normalization constant comparable to σ defined by

$$c_\sigma = \frac{1}{2(2 - \sigma)} \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+\sigma}} d\xi.$$

As in (8.4), if $(x, t) \in Q_{1/8}$, then we have that

$$\begin{aligned} & -(-\Delta^{\sigma/2})u(x, t) + (-\Delta^{\sigma/2})u(0, 0) \\ &= c_\sigma \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \varphi(y) \frac{2-\sigma}{|y|^{n+\sigma}} dy \\ & \quad + c_\sigma \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \phi(y) \frac{2-\sigma}{|y|^{n+\sigma}} dy \\ &= c_\sigma \left(Pu(x, t) - Nu(x, t) + \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \phi(y) \frac{2-\sigma}{|y|^{n+\sigma}} dy \right), \end{aligned}$$

where φ is the radial cut-off function in (8.4) and $\phi = 1 - \varphi$. Thus it follows from Lemma 8.1, Corollary 8.2 and (8.6) that

$$\sup_{K \in \mathcal{K}_2} |C_K^\phi(t)| \lesssim (|x|^\sigma + |t|)^\frac{\alpha}{\sigma} \|u\|_{L_T^\infty(L_\omega^1)},$$

and thus there is some $\alpha \in (0, 1)$ such that

$$(8.27) \quad |(-\Delta^{\sigma/2})u(x, t) - (-\Delta^{\sigma/2})u(0, 0)| \lesssim (|x|^\sigma + |t|)^\frac{\alpha}{\sigma} \|u\|_{L_T^\infty(L_\omega^1)}$$

for any $(x, t) \in Q_{1/8}$. Now, by Corollary 3.3, it is easy to check that

$$\begin{aligned} \mathbf{M}_2^-(\tau_x^t u - u)(0, 0) &\leq \partial_t u(x, t) - \partial_t u(0, 0) \\ &= \mathbf{I}u(x, t) - \mathbf{I}u(0, 0) \leq \mathbf{M}_2^+(\tau_x^t u - u)(0, 0). \end{aligned}$$

Thus, by Lemma 8.1 and Corollary 8.2, we have the estimate

$$\begin{aligned} (8.28) \quad |\partial_t u(x, t) - \partial_t u(0, 0)| &\leq |\mathbf{M}_2^-(\tau_x^t u - u)(0, 0)| \vee |\mathbf{M}_2^+(\tau_x^t u - u)(0, 0)| \\ &\leq \Lambda(Pu(x, t) + Nu(x, t)) \\ &\lesssim (|x|^\sigma + |t|)^\frac{\alpha}{\sigma} \|u\|_{L_T^\infty(L_\omega^1)} \end{aligned}$$

for any $(x, t) \in Q_{1/8}$. Hence by a standard translation argument of (8.27) and (8.28), and the remark (ii) below Theorem 2.1, we conclude that

$$\|u\|_{C^{\sigma+\alpha}(Q_{1/2})} \lesssim \|u\|_{L_T^\infty(L_\omega^1)}.$$

Therefore we complete the proof. \square

REFERENCES

- [AK] H. Abels and M. Kassmann, *An analytic proof to purely nonlocal Bellman equations arising in models of stochastic control*, J. Differential Equations **236**, 2007, 29–56.
- [CC] Luis A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, volume 43 of American Mathematical Society, Colloquium Publications. American Mathematical Society, Providence, RI, 1995.
- [CS1] Luis A. Caffarelli and Luis Silvestre, *The Evans-Krylov theorem for nonlocal fully nonlinear equations*, Annals of Math. **174**, 2011, 1163–1187.
- [CS2] Luis A. Caffarelli and Luis Silvestre, *On the Evans-Krylov theorem*, Proc. Amer. Math. Soc. **138**, 2010, 263–265.
- [Ev] L. C. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **35**, 1982, 333–363.
- [KL1] Yong-Cheol Kim and Ki-Ahm Lee, *Regularity results for fully nonlinear integro-differential operators with nonsymmetric positive kernels : Subcritical Case*, Potential Anal. **38**(2), 2013, 433–455.
- [KL2] Yong-Cheol Kim and Ki-Ahm Lee, *Regularity results for fully nonlinear integro-differential operators with nonsymmetric positive kernels*, Manuscr. Math. **139**, 2012, 291–319.
- [KL3] Yong-Cheol Kim and Ki-Ahm Lee, *Regularity results for fully nonlinear parabolic integro-differential operators*, Math. Ann. **357**, 2013, 1541–1576.

- [KL4] Yong-Cheol Kim and Ki-Ahm Lee, *Cordes-Nirenberg type estimates for nonlocal parabolic equations*, arXiv:1212.5591v4 [math.CA].
- [Kr] N. V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations*, Izv. Akad. Nauk SSSR Ser. Mat. **46**, 1982, 487–523.
- [MP] R. Mikulyavichyus and G. Pragarauskas, *Classical solutions of boundary value problems for some nonlinear integro-differential equations*, Liet. Mat. Rink. **34**, 1994, 347–361.
- [MR] J.-L. Menaldi and M. Robin, *Ergodic control of reflected diffusions with jumps*, Appl. Math. Optim. **35**, 1997, 117–137.

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